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AN INVARIANT FOR A SUBSPACE OF THE FINITE DIMENSIONAL VECTOR SPACE AND AUTOMORPHISM PARTITION OF A REAL SYMMETRIC MATRIX

N. S. CHAUDHARI, N. L. SARDA AND D. B. PHATAK

Department of Computer Science and Engineering, Indian Institute of Technology Bombay, Powai, Bombay 400076

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A reasonable concept for an automorphism of a real subspace of a given finite dimensional vector space $\mathbb{R}^n$ is introduced. Well-known concept of automorphism for graphs is generalized for real symmetric matrices. It is shown that there is a connection between these two concepts. To establish this connection, a matrix called "projection matrix" of a given subspace is used. Representation of a given subspace in terms of its projection matrix is shown to be unique. Two concepts, namely automorphism of a subspace and the automorphism of a real symmetric matrix (which is its projection matrix) have been shown to be isomorphic. Automorphisms of a given subspace induce automorphism partition. The automorphism partition induced by a given subspace is shown to be an invariant of the underlying subspace. It is also shown that, given a real symmetric matrix, its automorphisms can be expressed in terms of automorphisms of all of its eigenspaces. It is shown that certain computations on the adjacency matrices of strongly regular graphs as well as the matrix NN^T, where N is an incidence matrix of 2-PBIBD, are simplified.

1. INTRODUCTION

Automorphism concept for finite graphs is too well-known[2,7]. It is not very clear whether some interesting results can be obtained by generalizing this concept for square matrices of finite size. To investigate some consequences of such generalization, we first introduce a new combinatorial invariant, called "automorphism partition of subspace" for a subspace of finite dimensional vector space. Next, we show that this new invariant is in fact equivalent to an automorphism partition (in the generalized sense) of some symmetric matrix. This constitutes the main result of paper. We give a method to construct this symmetric matrix. In the examples, we point out (in the similar fashion as was done in Chaudhari *et al.*[1]) that certain computations required for obtaining such equivalent symmetric matrix are simplified if the underlying subspace is an eigenspace of some strongly regular graph.

We shall deal with only real vector spaces in this paper, although it is possible to generalize the main result for complex vector spaces.

2. Standard Notations from Graph Theory

The notations given below are well-known in graph theory and, in particular, we have taken them from Chaudhari *et al.*[5], Hoffmann[7] and Mathon[9].

An n-vertex simple graph, in which no multiple edges are permitted, is denoted by $G = (V, E)$. Here, V is the set of vertices and E in $V \times V$ is the set of edges. The cardinality of the set V is n (a finite natural number). An adjacency matrix of G is a binary symmetric square matrix of size n and is denoted by A. Without loss of generality, the set V can be chosen to be $\{1, 2, ..., n\}$.

An automorphism of a graph G is an incidence preserving bijection of its vertices V. Thus, if $V = \{1, 2, \ldots, n\}$, P is a permutation matrix of size n and A is an adjacency matrix of G, then a permutation (say p) of V represented by P is said to be an automorphism of G if and only if

$$A = P^{-1} AP. \qquad \ldots(2.1)$$

The set of all automorphisms of G forms the automorphism group Aut (G) of G. Automorphism group Aut (G) induces an automorphism partition π_A which partitions the set V. Two vertices x, y in V belong to the same cell of π_A if and only if $y = p(x)$ for some p in $Aut\,(G)$.

Let $\Pi\,(V)$ be the collection of all partitions of V. Given π_1, π_2 in $\Pi\,(V)$, we say that π_1 is finer than π_2, written as $\pi_1 < \pi_2$ if for every cell C_1 in π_1 there exists a cell C_2 in π_2 such that C_1 is a subset of C_2. With respect to this operation $<$, $\Pi\,(V)$ forms a lattice. The glb and lub of π_1 and π_2 are denoted by $\pi_1 \cap \pi_2$ and $\pi_1 \cup \pi_2$ respectively. It is easy to see that glb and lub are defined for any π_1 and π_2.

A 'coarsest' partition of V, consisting of only one cell containing all elements of V, is denoted by u. A graph G is said to be *transitive* if $\pi_A = u$. A special partition which is called as xth *basis* (x in V) is defined to be a two cell partition in which a vertex x forms one cell and the other cell is all the rest of the vertices in V; it is denoted by ϵ_x. A graph G is said to be *rigid* if π_A consists of n cells (each containing a single vertex), i. e.,

$$\pi_A = \epsilon_1 \cap \epsilon_2 \cap \ldots \cap \epsilon_n.$$

A graph G is *regular* if the degrees of all of its vertices are equal. A vertex z in V is a *neighbour* of a vertex pair (x, y) iff (x, z) and (z, y) are adjacent (i. e., edges in G). A graph G is 'strongly regular' with parameters (n, d, p, q) iff

(i) each vertex x in V has a degree d,

(ii) each edge (x, y) in E has p distinct neighbours,

(iii) each vertex pair (x, y) not in E has q distinct neighbours.

3. Some Notions for A Square Symmetric Matrix

Let M be a real, symmetric square matrix of size n. It may be noted that it is possible to generalize all the definitions in this section to a matrix whose elements are from any field and to nonsymmetric matrix. The set $V = \{1, 2, ..., n\}$ can be viewed to be an index set for M. In particular, we may define $f_M : V \times V \rightarrow \mathbb{R}$ that represents M.

An automorphism of a square matrix M is a bijection of its index set that preserves the matrix entries. Thus, if M is a given square matrix of size n and $V = \{1, 2, ..., n\}$ is the set used for indexing rows as well as columns of M, then we say that a bijection $p : V \rightarrow V$ is an automorphism of M (or equivalently, the function f_M) if and only if, for all i, j in V,

$$f_M (i, j) = f_M (p (i), (p (j)). \quad ...(3.1)$$

If P is a permutation matrix that represents p, then p is an automorphism M if and only if

$$M = P^{-1} MP. \quad ...(3.2)$$

For numerical purposes, considering the fact that the entries is M may never be represented exactly (eg., when stored on some storage device like memory of a digital computer), we can allow a small error ϵ in such representation and we define "ϵ-automorphism of M" if we replace the condition in (3.1) by the following :

$$| f_M (i, j) - f_M (p (i), p (j)) | \leqslant \epsilon. \quad ...(3.3)$$

The proofs of the following results are trivial and are on the lines similar to their counterparts in graph theory.

The set of all automorphisms of M forms an automorphism group Aut (M) of M. (The group operation is usual composition of bijections). The automorphism group Aut (M) induces the automorphism partition π_M which partition the set $V = \{1, 2, ..., n\}$. Two indices x and y in $\{1, 2, ..., n\}$ belong to the same cell of π_M if and only if $y = p (x)$ for some p in Aut (M).

4. An Invariant of a Subppace

Let $\mathbb{R}^n$ be the n-dimensional (n a finite natural number) vector space with the inner product defined as

$$<x, y> = \sum_{i=1}^{n} x_i \, y_i. \quad ...(4.1)$$

We shall take a vector to be a column vector unless stated otherwise.

Let $\mathcal{U}$ be an m-dimensional ($m \leqslant n$) subspace of $\mathbb{R}^n$. One of the representations of $\mathcal{U}$ is in terms of m linearly independent n-dimensional vectors. In many situations,

we are not interested in saying that $\mathscr{U}$ is isomorphic to $\mathbf{R}^m$, but the structure of the subspace $\mathscr{U}$ in $\mathbf{R}^n$ is of interest. In such cases, $\mathscr{U}$ can be completely specified by m linearly independent vectors say $u_1, u_2, \ldots, u_m$ that lie in $\mathscr{U}$. Thus $\mathscr{U}$ can be represented by an $m \times n$ reactangular matrix consisting of these linearly independent vectors. Let this be denoted by U. Thus,

$$\underset{(m \times n)}{U} = \begin{bmatrix} u_1^T \\ u_2^T \\ \vdots \\ u_m^T \end{bmatrix}. \qquad \ldots(4.2)$$

Our aim is to associate certain permutations of $\{1, 2, \ldots, n\}$ with the subspace $\mathscr{U}$ or equivalently with the matrix U. Let p be such a typical permutation, and the corresponding permutation matrix (of size $n \times n$) be P. It is possible to postmultiply U by P. To retain the structure of the subspace $\mathscr{U}$ in $\mathbf{R}^n$, we need to impose the condition that the subspace spanned by "UP" (i. e., m vectors that form rows of matrix UP) is the same as the subspace spanned by the rows of U. In order to formally express this notion in terms of matrices, we need the following lemma.

Lemma 4.1—Let an $m \times n$ matrix U consist of m linearly independent rows. Let P be an $n \times n$ permutation matrix. Then the matrix UP also consists of m linearly independent rows.

PROOF: Given that $u_1^T, u_2^T, \ldots u_m^T$ are linearly independent. This means that, if there exist m scalars $s_1, s_2, \ldots, s_m$ satisfying

$$s_1 u_1 + s_2 u_2 + \ldots + s_m u_m = 0 \qquad \ldots(4.3)$$

then

$$s_1 = s_2 = \ldots = s_m = 0.$$

Let us write down the constraints imposed by (4.3) on $s_1, s_2, \ldots, s_m$ explicitly in terms of scalar components of u_i's. For this purpose, let

$$\begin{aligned} u_1^T &= [u_{11}\ u_{12} \ldots u_{1n}] \\ u_2^T &= [u_{21}\ u_{22}, \ldots u_{2n}] \\ &\vdots \\ u_m^T &= [u_{m1}\ u_{m2} \ldots u_{mn}]. \end{aligned} \qquad \ldots(4.4)$$

Then, the vector equation (4.3) is equivalent to the following n equations that must be satisfied by $s_1, s_2, \ldots, s_m$:

$$\begin{aligned} s_1 u_{11} + s_2 u_{21} + \ldots + s_m u_{m1} &= 0 \\ s_1 u_{12} + s_2 u_{22} + \ldots + s_m u_{m2} &= 0 \\ &\vdots \\ s_1 u_{1n} + s_2 u_{2n} + \ldots + s_m u_{mn} &= 0. \end{aligned} \qquad \ldots(4.5)$$

Let P denote a permutation $p : V \to V$ where $V = \{1, 2, \ldots, n\}$. Next, let us consider the matrix UP. Its rows consist of $u_1^T P, u_2^T P, \ldots, u_m^T P$. Assume that there exist the scalars $s_1', s_2', \ldots, s_m'$, not all zero, such that

$$s_1' u_1^T P + s_2' u_2^T P + \ldots + s_m' u_m^T P = o^T. \qquad \ldots(4.6)$$

Now, it is possible to write down n equations that must be satisfied by $s_1', s_2' \ldots s_m'$, as:

$$\begin{aligned} s_1' u_{1p(1)} + s_2' u_{2p(1)} + \ldots + s_m' u_{mp(1)} &= 0 \\ s_1' u_{1p(2)} + s_2' u_{2p(2)} + \ldots + s_m' u_{mp(2)} &= 0 \\ &\vdots \\ s_1' u_{1p(n)} + s_2' u_{2p(n)} + \ldots + s_m' u_{mp(n)} &= 0. \end{aligned} \qquad \ldots(4.7)$$

Since p is a permuation from V to V, it is easy to see that the system of equations in (4.5) and (4. 7) are the same. Hence our assumption implies that $u_1, u_2 \ldots, u_m$ are linearly dependent, which is a contradiction. Hence the result. Q.E.D.

Returning to the two $m \times n$ matrices U and UP at our disposal, we have said that the rows of U consist of linearly independent n-dimensional (row) vectors. Hence, by the above lemma, the rows of UP also consist of linearly independent n-dimensional vectors. We would like to consider only those P's for which we "remain" in the same subspace $\mathscr{U}$ of $\mathbb{R}^n$. Since the rank of U and UP are both equal to m, this link between U and UP, namely, "remaining in the subspace $\mathscr{U}$ of $\mathbb{R}^n$", can be established by stipulating the existence of an $m \times m$ square 'nonsingular' matrix L such that

$$U = LUP.$$

Hence, we define the concept of automorphism of the subspace $\mathscr{U}$ of $\mathbb{R}^n$ (in the sense of preserving the 'structure' of the subspace $\mathscr{U}$ in $\mathbb{R}^n$) as follows:

Definition —Let the subspace $\mathscr{U}$ of $\mathbb{R}^n$ be represented by an $m \times n$ matrix U consisting of m linearly independent n-dimensional (row) vectors in $\mathbb{R}^n$. Let a permu-

tation $p : \{1, 2, \ldots. n\} \rightarrow \{1, 2, \ldots, n\}$ be denoted by the corresponding $n \times n$ permutation matrix P. We say that the permutation p is an 'automorphism of the subspace' $\mathscr{U}$ iff there exists a nonsingular $m \times m$ matrix L such that

$$U = LUP. \qquad \ldots(4.8)$$

It is possible to take any m linearly independent n-dimensional vectors, say v_1, v_2, ..., v_m that represent the subspace $\mathscr{U}$. The above concept would be of little value if the corresponding $m \times n$ matrix

$$\underset{(m \times n)}{V} = \begin{bmatrix} v_1^T \\ v_2^T \\ \vdots \\ v_m^T \end{bmatrix}$$

does not have the same set of permutations that are "eligible automorphisms of subspace" as defined via P matrix in (4.8). To emphasize that this is not the case, we state the following theorem.

Theorem 4.1—If there is an $m \times n$ matrix whose rows consist of m linearly independent n-dimensional (row) vectors u_1^T, u_2^T, ..., u_m^T, that represent the subspace $\mathscr{U}$ of $\mathbb{R}^n$ for which there exists permutation matrix P with the property that the existence of nonsingular L matrix of size $m \times m$ satisfying

$$U = LUP \qquad \ldots(4.9)$$

is guaranteed, then for any $m \times n$ matrix V whose rows consist of m linearly independent n-dimensional vectors v_1^T, v_2^T, .., v_m^T, that represents the same subspace $\mathscr{U}$ of $\mathbb{R}^n$, there exists a nonsingular $m \times m$ matrix L' satisfying

$$V = L'VP \qquad \ldots(4.10)$$

for the same P.

PROOF : Under the conditions given in the above theorem, it is easy to see that the matrices U and V are related as :

$$U = L_1 V$$

where L_1 is an $m \times m$ square nonsingular matrix. Substituting (4.11) in (4.9), we get

$$L_1 V = L\,(L_1 V)\,P.$$

Since L_1 is nonsingular, L_1^{-1} exists. So, by premultiplying both sides of the above equation by L_1^{-1} and applying associativity of matrix multiplication.

$$V = (L_1^{-1} LL_1) VP. \qquad \text{...(4.11)}$$

Thus, the theorem follows if we identify

$$L' = L_1^{-1} LL_1.$$

Q.E.D.

It easily follows that, for any vector v in $\mathscr{V}$, we have the result that vP also remains in the subspace $\mathscr{V}$, where P is the permutation matrix satisfying (4.9). Another interesting observation is that the set of permutations that we have defined to be automorphisms of the subspace $\mathscr{V}$ form a group under usual composition operation. To formally state this result, we give the following theorems.

Theorem 4.2—For an $m \times n$ matrix U with rank m ($m \leqslant n$) if there exists $n \times n$ permutation matrices P_1 and P_2 for which there are some nonsingular $m \times m$ matrices L_1 and L_2 satisfying

$$U = L_1 UP_1$$

$$U = L_2 UP_2 \qquad \text{...(4.12)}$$

then there exists on $m \times n$ nonsingular matrix L_3 satisfying

$$U = L_3\, U\, (P_1\, P_2). \qquad \text{...(4.13)}$$

PROOF : Given

$$U = L_1 UP_1 \qquad \text{...(4.12a)}$$

$$U = L_2\, UP_2. \qquad \text{...(4.12b)}$$

Substitute for U in the right hand side of (4.12b) from (4.12a).

$$U = L_2\, (L_1 UP_1)\, P_2$$

$$= (L_2\, L_1)\, U\, (P_1\, P_2).$$

Hence $L_3 = L_2\, L_1$ is the required nonsingular matrix.

Q.E.D.

Theorem 4.3—For an $m \times n$ matrix U with rank m ($m \leqslant n$), if there exists an $n \times n$ permutation matrix P for which there is some $m \times m$ nonsingular matrix satisfying

$$U = LUP \qquad \text{...(4.14)}$$

then there exists an $m \times m$ nonsingular matrix L' satisfying

$$U = L'UP^{-1}. \qquad \text{...(4.15)}$$

PROOF : Given

$$U = LUP \qquad \text{...(4.16)}$$

with L nonsingular. Noting that P is also nonsingular permultiply (4.14) by L^{-1} and postmultiply by P^{-1} to get

$$U = L^{-1}\, UP^{-1}.$$

Hence $L' = L^{-1}$ is the required nonsingular matrix.

Q.E.D.

It is trivial to see that an identity permutation acts as an identity element in the set of permutations that form automorphisms of a given subspace. Noting that the composition operation of two permutations is associative and the inverse of a given permutation always exists, it follows that the automorphisms of a given subspace indeed form a group. Let us denote this group by Aut (U). Now it is easy to see that the automorphism group Aut (U) induces an automorphism partition that we shall denote by π_U. π_U partitions the set $V = \{1, 2, \ldots, n\}$. (Consider the relation $R \subset V \times V$ such that i and j are related by R iff there exists a permutation p in Aut (U) such that $j = p\,(i)$. Transitivity follows from the composition of permutations being closed over the set Aut (U), i.e., Theorem 4.2. Symmetry follows from inverse of permutations being closed over Aut (U), i. e., Theorem 4.3. Reflexivity follows from the fact that an identity permutation is always in Aut (U).) This partition π_U is an invariant that we assign to the subspace $\mathscr{U}$ of $\mathbf{R}^n$.

We have chosen U matrix to be an $m \times n$ matrix of rank m with $m \leqslant n$. For the reactangular matrices of the type $n \times m$ with $m \leqslant n$ having rank m, the similar concept can easily be defined. In this case, the permutation matrix P will be postmultiplied and nonsingular matrix L will be postmultiplied in our basic eqn (4.8).

5. Automorphism Partition of a Subspace in Terms of Automorphism Partition of a Square Symmetric Matrix

In section 3, we introduced the concept of automorphism partition π_M of a square matrix. In the previous section, the concept of automorphism partition π_U of an $m \times n$ $(m \leqslant n)$ reactangular matrix U with rank m was introduced. The concept of automorphism partition π_U for a rectangular matrix U heavily depends on the concept of automorphism partition of the underlying subspace represented by rows (or columns if $m > n$) of given reactangular matrix. In particular, these two concepts, namely automorphism partition of an $n \times n$ square matrix denoted by π_M and an automorphism partition of $m \times n$ reactangular matrix U with rank m denoted by π_U do not coincide when $m = n$. So, the link between two concepts must be clearly established. In this

section, we proceed to show that, corresponding to an $m \times n$ matrix U, there exists another $n \times n$ square symmetric matrix say C whose automorphism partition (in the sense of square matrix) π_C is the same as the automorphism partition (in the sense of rectangular matrix) π_U. We also prove that, unlike the nonuniqueness of U matrix to represent the subspace $\mathscr{U}$, the C matrix is unique for a given subspace $\mathscr{U}$.

For this purpose, we would like to take an $m \times n$ reactangular matrix W that consists of m orthonormal vectors say $w_1^T, w_2^T, \ldots, w_m^T$ which span the subspace $\mathscr{U}$. From U matrix whose rows represent the linearly independent vectors $u_1^T, u_2^T, \ldots, u_m^T$, these vectors ($w_1^T$ ' s) can be obtained by simply orthonormalizing them. It may be noted that, our definition of inner product of two vectors x, y in $\mathbb{R}^n$ is :

$$<x, y> = \sum_{i=1}^{n} x_i y_i \qquad \ldots(5.1)$$

which has a property than $<x, y> = <y, x>$.

Let us construct a matrix W from m orthonormal vectors $w_1^T, w_2^T, \ldots, w_m^T$ that span the subspace $\mathscr{U}$ as follows :

$$\underset{(m\times n)}{W} = \begin{bmatrix} w_1^T \\ w_2^T \\ \vdots \\ w_m^T \end{bmatrix}. \qquad \ldots(5.2)$$

Let us write the columns of W as $c_1, c_2, \ldots c_n$. Hence

$$\underset{(m\times n)}{W} = [c_1\, c_2, \ldots c_n] \qquad \ldots(5.3)$$

where each of the $c_1, c_2, \ldots, c_n$ can be viewed as a vector in $\mathbb{R}^m$. In the vector space $\mathbb{R}^m$, their inner products can be defined in the same way as (5.1). Thus, if c_i, c_j are in $\mathbb{R}^m$, then

$$<c_i, c_j> = \sum_{k=1}^{m} (c_i)_k\, (c_j)_k. \qquad \ldots(5.4)$$

Let us now construct an $n \times n$ square matrix C as follows :

$$\underset{(n\times n)}{C} = \begin{bmatrix} <c_1, c_1> & <c_1, c_2> & \ldots & <c_1, c_n> \\ <c_2, c_1> & <c_2, c_2> & \ldots & <c_2, c_n> \\ \vdots & \vdots & & \vdots \\ <c_n, c_1> & <c_n, c_2> & \ldots & <c_n, c_n> \end{bmatrix}. \qquad \ldots(5.5)$$

It may be noted that there is a slight difference between gram matrix[13] of an $m \times n$ $(m \leqslant n)$ matrix and the C matrix we have constructed above. Normally, we take the inner product in R^n $(n \geqslant m)$ to get gram matrix. Also, we construct gram matrix to inferthing s like linear dependence (or independence) of vectors as in Bellman[1], Myskis[10] and Voyevodin[13]. In this context, $m \times m$ matrix called gram matrix of an $m \times n$ matrix W is defined as :

$$\underset{(m\times n)}{G} = \underset{(m\times n)}{W} \underset{(n\times m)}{W^T} \quad (m \leqslant n). \qquad \ldots(5.6)$$

The notion of gram matrix essentially corresponds to taking inner product in that vector space from which the underlying vectors of interest are taken. In our case, these vectors can be viewed as n-dimensional row vectors that form rows of W. However, for the C matrix that we have defined in eqn. (5.5), we can write a compact equation as :

$$\underset{(n\times n)}{C} = \underset{(n\times m)}{W^T} \underset{(m\times n)}{W}. \qquad \ldots(5.7)$$

It may further be noted that the definition of G matrix as given in (5.6) is not of much interest, because, due to mutually orthonormal nature of vectors forming rows of W matrix, we have

$$\underset{(m\times n)}{G} = \underset{(m\times n)}{W} \underset{(n\times m)}{W^T} = \underset{(m\times m)}{I}. \qquad \ldots(5.8)$$

From the definition (5.4), we have $<c_i, c_j> = <c_j, c_i>$, and hence the matrix C is symmetric. We shall refer to C matrix as "projection matrix".

A nice geometric interpretation can be given to C-matrix of the subspace $\mathscr{U}$. It is given in the following theorem.

Theorem 5.1—Suppose that we want to project a vector x in R^n in the subspace $\mathscr{U}$. Then the component of x in $\mathscr{U}$ is given by Cx, where C is the matrix as defined above.

PROOF : We shall use linearity of the projection operator. This linearity is obvious from the bilinearity of inner product.

Let us first consider the projections of unit vectors $e_1, e_2, \ldots, e_j, \ldots, e_n$, where jth component of e_j is one and rest of the components are zero. Let us denote m orthonormal vectors that span $\mathscr{U}$ by $w_1, \ldots, w_m$.

Projection of e_j in the subspace $\mathscr{U}$, say $Pr(e_j)$ is given by,

$$Pr(e_j) = (w_1)_j w_1 + (w_2)_j w_2 + \ldots + (w_m)_j w_m \qquad \ldots(5.9)$$

$$= \begin{bmatrix} (w_1)_j (w_1)_1 + (w_2)_j (w_2)_1 + \ldots + (w_m)_j (w_m)_1 \\ (w_1)_j (w_1)_2 + (w_2)_j (w_2)_2 + \ldots + (w_m)_j (w_m)_2 \\ \vdots \\ (w_1)_j (w_1)_n + (w_2)_j (w_2)_n + \ldots + (w_m)_j (w_m)_n \end{bmatrix}. \qquad \ldots(5.10)$$

This can be expresed in terms of c_i's (defined by (5.2), (5.3)) as :

$$\begin{bmatrix} <c_j, c_1> \\ <c_j, c_2> \\ \vdots \\ <c_j, c_n> \end{bmatrix}. \qquad \text{...(5.11)}$$

Since j is arbitrary, we have

$$Pr\,(e_j) = \begin{bmatrix} <c_j, c_1> \\ <c_j, c_2> \\ \vdots \\ <c_j, c_n> \end{bmatrix}$$

for any $j = 1, 2, ..., n.$ ...(5.12)

It may be noted that C-matrix can now be written as :

$$C = [Pr\,(e_1)\; Pr\,(e_2)\; ...\; Pr\,(e_n)]. \qquad \text{...(5.13)}$$

Next, let us consider any vector x in $\mathbb{R}^n$

$$x = \sum_{i=1}^{n} x_i\, e_i. \qquad \text{...(5.14)}$$

Let $Pr\,(x)$ denote the projection of a vector x in the given subspace. Then

$$\begin{aligned} Pr\,(x) &= Pr\,[\sum_{i=1}^{n} x_i\, e_i] \\ &= \sum_{i=1}^{n} x_i\, Pr\,(e_i) \\ &= [Pr\,(e_j)\; \dots\; Pr\,(e_n)] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \\ &= Cx. \end{aligned} \qquad \text{...(5.15)}$$

Q.E.D.

From the above theorem, it may easily be verified that the projection matrix has exactly two eigenvalues, namely one and zero. The multiplicity of eigenvalue one is m, the dimension of subspace $\mathscr{U}$ for which projection matrix C is constructed; the multiplicity of eigenvalue zero is $n - m$. The eigenspace corresponding to eigenvalue one is $\mathscr{U}$, and the eigenspace corresponding to eigenvalue zero is the subspace spanned by the vectors orthogonal to all the vectors in $\mathscr{U}$.

Let us now relate the automorphisms of the subspace $\mathcal{U}$ to the automorphisms of its projection matrix.

Theorem 5.2. – Let $\mathcal{U}$ be a subspace of $\mathbf{R}^n$ with dimension m. Let U be an $m \times n$ rectangular matrix whose rows consist of m linearly independent vectors of $\mathbf{R}^n$ which span the whole of $\mathcal{U}$. Then a permutation p is in Aut (U) if and only if p is in Aut(C), where C is square matrix of size n constructed as above (eqn. 5.5).

PROOF : Let us concentratre on the subspace $\mathcal{U}^{\perp}$ which consists of all vectors orthogonal to the vectors in $\mathcal{U}$ (and also contains o vector as required by the definition of subspace; thus $\mathcal{U} \cap \mathcal{U}^{\perp} = o$). Let $W^{\perp}$ be an $(n - m) \times n$ matrix

$$\underset{((n-m)\times m)}{W^{\perp}} = \begin{bmatrix} (w_1^{\perp})^T \\ (w_2^{\perp})^T \\ \vdots \\ (w_{n-m}^{\perp})^T \end{bmatrix} \qquad \ldots(5.16)$$

of orthonormal vectors $(w_1^{\perp})^T, \ldots, (w_{n-m}^{\perp})^T$ which span $\mathcal{U}'^{\perp}$. If we consider all the n vectors, $w_1^T, \ldots, w_m^T$; $(w_1^{\perp})^T, \ldots, (w_{n-m}^{\perp})^T$, then they are orthonormal to each other, and the span the whole space $\mathbf{R}^n$.

Consider the matrix

$$\underset{(n\times n)}{X^T} = \begin{bmatrix} (w_1)^T \\ (w_2)^T \\ \vdots \\ (w_m)^T \\ (w_1^{\perp})^T \\ \vdots \\ (w_{n-m})^T \end{bmatrix}. \qquad \ldots(5.17)$$

The matrix X is clearly nonsingular. In fact it is an orthonormal matrix, hence $X^{-1} = X^T$. We can write $((n - m) \times m)$ matrix $W^{\perp}$ as :

$$\underset{((n-m)\times m)}{w^{\perp}} = [c_1^{\perp} \;\; c_2^{\perp} \;\; \ldots \;\; c_n^{\perp}] \qquad \ldots(5.18)$$

where each of the $c_1^{\perp}, c_2^{\perp}, \ldots c_n^{\perp}$ can be viewed as a vector in $\mathbf{R}^{(n-m)}$. In the vector

space $R^{(n-m)}$, their inner products can be defined as in (5.1). Thus, if $c_i^{\perp}$, $c_j^{\top}$ are in $R^{(n-m)}$, then

$$<c_i^{\perp}, c_j^{\perp}> = \sum_{k=1}^{n-m} (c_i^{\perp})_k \, (c_j^{\perp})_k. \qquad \ldots(5.19)$$

We can now construct a square matrix $C^{\perp}$ of size n as follows :

$$C^{\perp} = \begin{bmatrix} <c_1^{\perp}, c_1^{\perp}> & <c_1^{\perp}, c_2^{\perp}> & \ldots & <c_1^{\perp}, c_n^{\perp}> \\ <c_2^{\perp}, c_1^{\perp}> & <c_2^{\perp}, c_2^{\perp}> & \ldots & <c_2^{\perp}, c_n^{\perp}> \\ \vdots & & & \\ <c_n^{\perp}, c_1^{\perp}> & <c_n^{\perp}, c_2^{\perp}> & \ldots & <c_n^{\perp}, c_n^{\perp}> \end{bmatrix}. \qquad \ldots(5.20)$$

Now let us consider the matrix product $X^T X$. The matrix X^T can be written as :

$$X^T = \begin{bmatrix} c_1 & c_2 & \ldots & c_n \\ \ldots & \ldots & \ldots & \ldots \\ c_1^{\perp} & c_2^{\perp} & \ldots & c_n^{\perp} \end{bmatrix} \qquad \ldots(5.21)$$

Hence

$$XX^T = I = \begin{bmatrix} c_1^T & (c_1^{\perp})^T \\ c_1^T & (c_2^{\perp})^T \\ \vdots & \\ c_n^T & (c_n^{\perp})^T \end{bmatrix} \begin{bmatrix} c_1 & c_2 & \ldots & c_n \\ \ldots & \ldots & \ldots & \ldots \\ c_1^{\perp} & c_2^{\perp} & \ldots & c_n^{\perp} \end{bmatrix} \qquad \ldots(5.22)$$

$$= \begin{bmatrix} c_1^T c_1 + (c_1^{\perp})^T (c_1^{\perp}) & c_1^T c_2 + (c_1^{\perp})^T (c_2^{\perp}) & \ldots & c_1^T c_n + (c_1^{\perp})^T (c_n^{\perp}) \\ \vdots & \vdots & & \vdots \\ c_n^T c_1 + (c_n^{\perp})^T (c_n^{\perp}) & c_n^T c_2 + (c_n^{\perp})^T (c_2^{\perp}) & \ldots & c_n^T c_n + (c_n^{\perp})^T (c_n^{\perp}) \end{bmatrix} \qquad \ldots(5.23)$$

$$= \begin{bmatrix} <c_1, c_1> + <c_1^{\perp}, c_1^{\perp}> & <c_1, c_2> + <c_1^{\perp}, c_2^{\perp}> & \ldots & <c_1, c_n> + <c_1^{\perp}, c_n^{\perp}> \\ \vdots & & & \vdots \\ <c_n, c_1> + <c_n^{\perp}, c_1^{\perp}> & <c_n, c_2> + <c_n^{\perp}, c_2^{\perp}> & \ldots & <c_n, c_n> + <c_n^{\perp}, c_n^{\perp}> \end{bmatrix}$$

(*equation continued on p. 230*)

$$= C + C^{\perp} = I. \quad \ldots(5.24)$$

Let p be a permutation in Aut (U). Let P be the corresponding permutation matrix. Then there exists a nonsingular $m \times m$ square matrix L satisfying

$$W = LWP. \quad \ldots(5.25)$$

The property of orthonormality of the rows is also retained by the postmultiplication by a permutation matrix P. The proof of this property is similar to that of Lemma 4.1 and hence is omitted.

What is of interest to us is that the m orthonormal vectors represented by the rows of WP also span the same subspace $\mathscr{U}$. Let us denote $W' = WP$ and define the corresponding C' matrix. Thus,

$$W' = WP \quad \ldots(5.26)$$

$$= [c_1\ c_2\ \ldots\ c_n]\ P. \quad \ldots(5.27)$$

Also, we have,

$$W' = [c_1'\ \ c_2'\ \ \ldots\ \ c_n'\ ]$$

$$= [c_{p(1)}\ c_{p(2)}\ \ldots\ c_{p(n)}]. \quad \ldots(5.28)$$

Hence the corresponding C' matrix is

$$C' = (W')^T\ W'$$

$$= P^T\ W^T\ WP$$

$$= P^{-1}\ CP \quad \ldots(5.29)$$

$$= \begin{bmatrix} <c_{p(1)}, c_{p(1)}> & <c_{p(1)}, c_{p(2)}> & \ldots & <c_{p(1)}, c_{p(n)}> \\ & \vdots & & \\ <c_{p(n)}, c_{p(1)}> & <c_{p(n)}, c_{p(2)},> & \ldots & <c_{p(n)}, c_{p(n)}>. \end{bmatrix} \quad \ldots(5.30)$$

Now it needs to be proved that $C' = C$. For this purpose, the basic assumption available for us is given in (5.25), namely the existence of a nonsingular matrix L satisfying

$$W = LWP. \quad \ldots(5.25)$$

Since the rows of WP are in $\mathscr{U}$, and they span the whole of $\mathscr{U}$, we shall choose to represent $\mathscr{U}$ by m orthonormal row vectors of the matrix $W' = WP$. The subspace $\mathscr{U}^{\perp}$ can be represented by $W^{\perp}$. We compute C-matrices from W' and $W^{\perp}$. Thus, we can write the equation similar to (5.24) as :

$$I = C' + C^{\perp} \quad \ldots(5.31)$$

$$\Rightarrow P^{-1}\ CP + C^{\perp}. \quad \ldots(5.32)$$

In particular, subtracting $C^{\perp}$ from both sides, we get

$$P^{-1} CP = I - C^{\perp}. \qquad \ldots(5.33)$$

The similar equation from (5.24) for C matrix is

$$C = I - C^{\perp}. \qquad \ldots(5.34)$$

Hence,

$$C = P^{-1} CP. \qquad \ldots(5.35)$$

Thus, we have proved that P corresponds to an automorphism p in Aut (C).

To prove the converse part, let us assume that p is a permutation represented by a permutation matrix P such that

$$C = P^{-1} CP \qquad \ldots(5.36)$$

where C is an $n \times n$ projection matrix (note the difference between "gram matrix" and "projection matrix") as defined in eqn. (5.7) for some subspace $\mathscr{U}$. We first prove that, for any matrix $C^{\perp}$ obtained for the subspace $\mathscr{U}^{\perp}$, we have

$$C^{\perp} = P^{-1} C^{\perp} P \qquad \ldots(5.37)$$

To prove this, we need only to observe that

$$I = C + C^{\perp}.$$

If C satisfies (5.36) then by substituting for C from (5.36), we get

$$I = P^{-1} CP + C^{\perp}$$

$$I - P^{-1} CP = C^{\perp}$$

$$P^{-1} (I - C) P = C^{\perp}$$

$$P^{-1} C^{\perp} P = C^{\perp}.$$

Next, we shall construct that matrix

$$M = \lambda_1 C + \lambda_2 C^{\perp} \qquad \ldots(5.38)$$

where λ_1 and λ_2 are two 'distinct' scalars.

From (5.36) and (5.37) it immediately follows that

$$M = P^{-1} MP \qquad \ldots(5.39)$$

thereby implying that p is in Aut (M). Now, let p be in Aut (M). It is easy to see that M is a real symmetric square matrix of size n. By the spectral theory of real symmetric matrices[10,13] λ_1 corresponds to its one eigenvalue and λ_2 corresponds to its other eigenvalue. This is because, the subspace $\mathscr{U}$ can be identified as an eigenspace of the matrix M corresponding to the eigenvalue λ_1 (i. e., null space of the operator $M - \lambda_1 I$) and

the subspace $\mathscr{U}^{\perp}$ can be identified as an eigenspace of the matrix M corresponding to the eigenvalue λ_2 (i. e., null space of the operator $M - \lambda_2 I$). By the spectral theory for real symmetric matrices, it is well known that the decomposition of a given real symmetric square matrix in terms of its real eigenvalues and the corresponding eigenspaces is unique[1,10]. Also, the (real) eigenvalues and corresponding eigenspaces completely characterize (specify) the given matrix M. Thus M can be equivalently represented by its eigenvalues and corresponding eigenspaces.

Let us represent M in terms of its eigenvalues and eigenvectors. From the above discussion, we can choose to represent eigenspace corresponding to λ_1 by W of size $m \times n$ and that corresponding to λ_2 by $W^{\perp}$ of size $(n - m) \times n$.

$$M = S^T DS \qquad \text{...(5.40)}$$

$$= \lambda_1 W^T W + \lambda_2 (W^{\perp})^T W^{\perp}. \qquad \text{.. (5.41)}$$

Our axiom is that p is a permutation representing permutation matrix P such that

$$C = P^{-1} \ CP. \qquad \text{...(5.36)}$$

From this for a matrix M defined by (5.31), we have,

$$M = P^{-1} MP. \qquad \text{...(5.39)}$$

This, in turn implies, from (5.41), that

$$M = \lambda_1 (WP)^T (WP) + \lambda_2 (W_{\top} P)^T (W^{\perp} P). \qquad \text{...(5.42)}$$

From the theory of eigenspaces of the real symmetric matrices, eqn. (5.42) can be satisfied if and only if the rows of matrices WP and $W^{\perp} P$ span the subspace $\mathscr{U}$ and $\mathscr{U}^{\perp}$ which are eigenspaces of M corresponding to eigenvalues λ_1 and λ_2 respectively. Since the rows of W as well as WP consist of m orthonormal vectors, and by (5.42) they span the same subspace $\mathscr{U}$, there must exist a nonsingular $m \times n$ matrix L such that

$$W = LWP. \qquad \text{...(5.43)}$$

Q.E.D.

In the passing, we note that if the subspace $\mathscr{U}$ of $\mathbf{R}^n$ is taken to be $\{0\}$ or $\mathbf{R}^n$, then the automorphism partition of this subspace is a single partition consisting of all element of $V = \{1, 2; \ldots, n\}$ which is 'coarsest' partition of V denoted by u. However, it may be noted that these are not the only subspaces which have the automorphism partition as u. For example, consider $\mathscr{U}$ to be spanned by vector $(1, 1, \ldots, 1)^T$ in $\mathbf{R}^n$ (or, the $n - 1$ diemensional subspace $\mathscr{U}^{\perp}$). It is easy to see that this subspace also has an automorphism partition u. For still more nontrivial example, consider the eigenspace of a transitive strongly regular graph (n, d, p, q) corresponding to any eigenvalue not equal to d. One set of such parameters n, d, p, q for which transitive strongly regular graph exists is (25, 12, 5, 6) and is given in Corneil *et al*[6]. In section 7.4, we shall see the adjacency listing of this graph. It follows that the eigenspace for eigenvalue $\lambda_1 = 2$ or

the eigenspace for eigenvalue $\lambda_2 = -3$ associated with the adjacency matrix of a first strongly regular graph given in section 7.4 will necessarily have automorphism partition as u. It may further be noted that, for an eigenvalue equal to d, corresponding eigenspace is that spanned by $(1, 1, \ldots, 1)^T$ in $\mathbb{R}^n$ and for this space automorphism partition is u. These subspaces, for which automorphism partition is u, may be called as 'transitive' subspaces; in fact many notions from group theory and graph theory can now be naturally introduced for subspaces.

The concept of obtaining the information about a subspace via construction of the matrix M such that

$$M = \lambda_1 C + \lambda_2 C^{\perp} \; (\lambda_1 \neq \lambda_2) \qquad \ldots(5.38)$$

is very useful. It may be noted that, the above representation of a real symmetric M is actually spectral decomposition of M, which has exactly 2 distinct eigenvalues λ_1 and λ_2. The above representation suggests one neat way to obtain the subspace $\mathscr{U}$ for which C matrix was constructed: namely, $\mathscr{U}$ is the eigenspace of real symmetric M corresponding to eigenvalue λ_1. In a similar fashion, the subspace $\mathscr{U}^{\perp}$ can be obtained by considering the eigenspace of M corresponding to eigenvalue λ_2.

We have already said that a permutation p in Aut (C) implies that p is in Aut(M). The converse, namely if p is a permutation in Aut (M) and M has two distinct eigenvalues λ_1 and λ_2 with spectral resolution given by (5.38), then p is in Aut (C), can now easily be proved.

We note that the representation of a given subspace in the terms of C matrix is unique. To emphasize nonuniqueness of other representations of $\mathscr{U}$, we cite two representataions. namely in terms of U matrix of size $m \times n$ whose rows consist of linearly independent m vectors which span the whole of $\mathscr{U}$, and another in terms of W matrix of size $m \times n$ whose rows consist of orthonormal m vectors which span the whole of $\mathscr{U}$. It may be noted that, we may take any m linearly independent vectors for forming U matrix, or any m orthonormal vectors for forming W matrix, provided these vectors are in $\mathscr{U}$; this leads to nonuniqueness of U or W matrices for subspace representation.

To emphaize the uniqueness of representation of $\mathscr{U}$ in terms of C matrix, we point out first that given $\mathscr{U}$, we can obtain W matrix and from that matrix construct C matrix as follows :

$$\underset{(n\times n)}{C} = \underset{(n\times m)}{W^T} \; \underset{(m\times n)}{W} \qquad \ldots(5.7)$$

To prove the reverse part, we first investigate whether there is an elegant method to construct $\mathscr{U}$ from C. It is indeed possible to uniquely get back $\mathscr{U}$ from C. For this this purpose, we introduce the following theorem:

Theorem 5.3—The subspace $\mathscr{U}$ is exactly the same subspace spanned by the rows of C. (rows of C = columns of C, since C is symmetric).

To prove this theorem, let us first state the following lemma.

Lemma 5.1—The vectors formed by rows of C lie in the subspace $\mathscr{U}$.

PROOF : Consider the ith row of C. It is given by:

$$i\text{th row of } (W^T W) = W^T \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w^m \end{bmatrix}$$

$$= s_1 w_1 + s_2 w_2 \ldots + + s_m w_m \qquad \ldots(5.44)$$

where $s_1 = w_{i1}^T$, $s_2 = w_{i2}^T$, .. , $s_m = w_{im}^T$

which is a linear combination of vectors which lie in $\mathscr{U}$. Since i is arbitrary, the result in the lemma follows.

Next, since W has rank m, it follows that there exists an $m \times n$ submatrix in W, which is nonsingular. Hence, there exist m rows of C which can be expressed as rows of LW, where L is an $m \times m$ nonsingular matrix. Hence, it follows that there exist m rows of C which are linearly independent. From this, and Lemma 5.1, it is easy to see that the vectors formed by rows (or columns) of C span exactly the whole of the subspace $\mathscr{U}$.

.E.D.

We have not yet proved the uniqueness of representation of $\mathscr{U}$ in terms of C matrix. For this purpose, consider the subspace $\mathscr{U}^\perp$ and choose $(n - m)$ orthonormal vectors that span $\mathscr{U}^\perp$ to form a matrix $W^\perp$ and hence $C^\perp$. Now from (5.24), we have

$$C = I - C^\perp. \qquad \ldots(5.24)$$

Let, if possible, there be a matrix W' whose rows are mutually orthonormal and span exactly the whole of $\mathscr{U}$, such that C' is not equal to C. To represent $\mathscr{U}^\perp$, however we can use $W^\perp$, and hence $C^\perp$. This forces C' to

$$C' = I - C^\perp. \qquad \ldots(5.45)$$

Thus we get $C = C'$ which is contradictory to our assumption. Hence the result follows by contraposition. The matrix C can thus be viewed to be complete invariant of the subspace $\mathscr{U}$.

5. AUTOMORPHISM PARTITION OF A REAL SYMMETRIC MATRIX IN TERMS OF AUTOMORPHISM PARTITIONS OF ITS EIGENSPACES

A real symmetric square matrix M can be expressed by spectral resolution as :

$$M = \lambda_1 w_1 w_1^T + \lambda_2 w_2 w_2^T + \ldots + \lambda_n w_n w_n^T \qquad \ldots(6.1)$$

where w_i's are mutually orthonormal n eigenvectors corresponding to (real) eigenvalues λ_i. In particular, if there are $\lambda_1, \lambda_2, \ldots \lambda_d$ distinct eigenvalues with multiplicities $m_1, m_2, \ldots, m_d$ then we can write M in terms of C_j matrices of section 5 as follows :

$$M = \sum_{j=1}^{d} \lambda_j C_j \qquad \ldots(6.2a)$$

where

$$C_j = \sum_{\substack{w_k \text{ is an eigenvector} \\ \text{corresponding to} \\ \text{eigenvalue } \lambda_j}} w_k w_k^T . \qquad \ldots(6.2b)$$

The discussion is section 5 indicates that C_j matrices contain full information about the eigenspace assigned to eigenvalue λ_j. In partiuular, the subspace spanned by the rows of C_j is precisely the eigenspace assigned to the eigenvalue λ_j. The concept of automorphism partition of C_j matrix is equivalent to the concept of automorphism partition of underlying subspace. The question that we investigate is, given a general real symmetric matrix M, is it possible to express the automorphism of M in terms of the automorphisms of certain subspaces. A special case of this problem when M has exactly two distinct eigenvalues has already been discussed in the previous section.

Theorem 6.1—A permutation p of n numbers is in Aut (M) if and only if, for all $j = 1, 2, \ldots, m$, p is in Aut (C_j).

Proof : Let p be in Aut (M). Let P be the corresponding permutation matrix such that

$$M = P^{-1} MP. \qquad \ldots(6.3)$$

Substituting for M from (6.2) we get.

$$M = \sum_{j=1}^{d} \lambda_j C_j = P^{-1} \left(\sum_{j=1}^{d} \lambda_j C_j \right) P$$

$$= \lambda_k P^{-1} C_k P + P^{-1} \left(\sum_{\substack{j=1 \\ j \neq k}}^{d} \lambda_j C_j \right) P \qquad \ldots(6.4)$$

where k is arbitrarily chosen. Note that the eigenspaces of a given real symmetric matrix M are unique. So, we may choose any m_j orthonormal vectors in jth eigenspace (corresponding to λ_j) and form C_j matrices. Let us choose to obtain C_j matrices in this fashion for all j except for $j = k$. Then eqn. (6.4) becomes

$$M = \sum_{j=1}^{d} \lambda_j C_j = \lambda_k P^{-1} C_k P + \sum_{\substack{j=1 \\ j \neq k}}^{d} \lambda_j C_j .$$

Hence,

$$C_k = P^{-1} C_k P. \quad \text{.. (6.5)}$$

Hence p is an automorphism of C_k. Since k is arbitrary, p is an automorphism for all C_k's ($k = 1, 2, \ldots, d$).

To prove the converse part, let p be in Aut (C_j) for all $j = 1, 2, \ldots, d$. Thus there exists a permutation matrix P (corresponding to p) such that

$$C_j = P^{-1} C_j P \text{ for all } j = 1, 2, \ldots, d. \quad \text{...(6.6)}$$

Substituting this in (6.2), we get

$$M = \sum_{j-1}^{d} \lambda_j (P^{-1} C_j P)$$

$$= P^{-1} \left(\sum_{j-1}^{d} \lambda_j C_j \right) P$$

$$= P^{-1} M P.$$

Hence p is in Aut (M). Q.E.D.

Theorem 6.2—p is in Aut (C_j) for all $j = 1, 2, \ldots, d$ if and only if p is in Aut (C_j), for $j = 1, 2, \ldots, k-1, k+1, \ldots, d$ (i.e., k is left out) for any arbitrary k, where C_j matrices correspond to eigenspaces of some real symmetric matrix M and are defined in (6.2).

The proof of this theorem follows easily from the fact that the eigenspace corresponding to λ_k is orthogonal to all the remaining eigenspaces. Thus, the proof is on the similar lines to that we have given for proving (5.30) from (5.29) in the proof of Theorem 5.1.

The spectral resolution of real symmetrix M is given below:

$$M = \sum_{j-1}^{d} \lambda_j C_j. \quad \text{...(6.2)}$$

Although there are many sets of orthonormal vectors (each set consisting of m_j vectors) that represent the eigenspace corresponding to eigenvalue λ_j, the symmetric $n \times n$ matrix C_j is unique. Another property of C_j matrices is :

$$\sum_{j=1}^{d} \underset{(n\times n)}{C_j} = \underset{(n\times n)}{I}. \quad \text{...(6.7)}$$

Thus, it may be noted that the real symmetric matrix M can be viewed as a linear combination of C_j matrices whose sum is unity matrix. These weightages correspond to the eigenvalues (λ_j) for the corresponding C_j matrices.

Using the notations introduced in section 2, it follows from Theorem 6.1 that,

$$\pi_M = \pi_{C_1} \cap \pi_{C_2} \cap \ldots \pi_{C_d} \qquad \ldots(6.8)$$

$$= \pi_{U_1} \cap \pi_{U_2} \cap \ldots \cap \pi_{U_d} \qquad \ldots(6.9)$$

where C_j's have already been defined. U_j matrices correspond to $m_j \times n$ matrices whose rows contain m_j linearly independent n-dimensional (row) vectors in $\mathbb{R}^n$ whicn span the eigenspace corsesponding to λ_j.

From the above discussion, it immediately follows that, corresponding to a given real symmetric matrix M, there exists a symmetric positive definite matrix say, F, with the property that Aut (M) = Aut (F); the difference between the distinct eigenvalues of the matrix F can be any arbitrary nonzero number.

7. Examples

7.1 *Latin Squares*

Let us consider an automorphism of the subspace spanned by the columns of the following matrix based on Latin Square :

$$\underset{(2n\times n)}{U^T} = \begin{bmatrix} \underset{(n\times n)}{I} \\ \underset{(n\times n)}{(LS)} \end{bmatrix} \qquad \ldots(7.1)$$

where $\underset{(n\times n)}{(LS)}$ denotes a Latin Square[12] of size n on symbols $\{1, 2, \ldots, n\}$. In particular, let us take a concrete example of $\underset{(3\times 3)}{(LS)}$ as

$$\underset{(3\times 3)}{(LS1)} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{bmatrix}. \qquad \ldots(7.2)$$

Then,

$$\underset{(6\times 3)}{U^T} = \begin{array}{c} a \\ b \\ c \\ d \\ e \\ f \end{array} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{bmatrix} \qquad \ldots(7.3)$$

Then there exists a permutation which takes the rows $a \to b$, $b \to c$, $c \to a$, $d \to f$, $e \to d$, and $f \to e$ which, when applied to the above matrix, gives us the following matrix :

$$
= \begin{array}{c} \\ a \\ b \\ c \\ d \\ e \\ f \end{array}
\begin{array}{c}
\begin{array}{cccccc} a & b & c & d & e & f \end{array} \\
\left[\begin{array}{cccccc}
0 & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 0
\end{array}\right]
\end{array}
\left[\begin{array}{ccc}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
1 & 2 & 3 \\
2 & 3 & 1 \\
3 & 1 & 2
\end{array}\right]
\qquad \ldots(7.4) \quad \ldots(7.5)
$$

$$
= \left[\begin{array}{ccc}
0 & 0 & 1 \\
1 & 0 & 0 \\
0 & 1 & 0 \\
2 & 3 & 1 \\
3 & 1 & 2 \\
1 & 2 & 3
\end{array}\right]
$$

$$
= P^T U^T. \qquad \ldots(7.6)
$$

It is easy to see that we remain in the same subspace spanned by columns of U^T under the application of this permutation. It would be interesting to obtain general results for subspaces based on Latin squares and formulated through equations similar to (7.1). In particular, it would be interesting to investigate whether the automorphism partition of such subspaces consists of exactly two partitions.

7.2 *Strongly Regular Graphs*

It is well-known[2] that a strongly regular graph (n, d, p, q) has exactly three distinct eigenvalues. One of the eigenvalue is $\lambda_0 = d$, with multiplicity $m_0 = 1$, and the corresponding eigenvector in $\mathbb{R}^n$ is given by $(1, 1, \ldots, 1)^T$. The remaining two eigenvalues are given by[2] :

$$\lambda_1 = (1/2)\{p - q + [(p - q)^2 + 4\,(d - q)]^{1/2}\} \qquad \ldots(7.7a)$$

$$\lambda_2 = (1/2)\{p - q - [(p - q)^2 + 4\,(d - q)]^{1/2}\}. \qquad \ldots(7.7b)$$

Their multiplicities are given by :

$$m_1 = (1/2)\{(n - 1) + [(n - 1)(q - p) - 2d]\,[(p-q)^2 + 4\,(d - q)]^{\,1/2}\} \qquad \ldots(7.8a)$$

$$m_2 = (1/2)\{(n - 1) - [(n - 1)(q-p) - 2d]\,[(p - q)^2 + 4\,(d - q)]^{-1/2}\} \qquad \ldots(7.8b)$$

It may be noted that the values of λ_0, λ_1, λ_2, m_0, m_1, m_2 are fixed by the parameters (n, d, p, q) of a given strongly regular graph. In particular, they are same for nonisomorphic strongly regular graphs with the same parameters (n, d, p, q). However, the

difference arising due to nonisomorphism is reflected into the corresponding eigenspaces (or equivalently C_j matrices). In this section, we give certain properties of the eigenspaces associated with eigenvalues λ_1, λ_2 given above (or, equivalently, properties of C_1, C_2 matrices).

Let the adjacency matrix of a strongly regular graph (n, d, p, q) (a symmetric binary square matrix of size n) be denoted by A. By spectral resolution, A can be expressed as :

$$A = d \underset{(n \times n)}{\begin{bmatrix} 1/n & 1/n & 1/n \\ 1/n & 1/n & 1/n \\ \vdots & \vdots & \vdots \\ 1/n & 1/n & 1/n \end{bmatrix}} + \lambda_1 C_1 + \lambda_2 C_2$$

$$= (d/n)\, J + \lambda_1 C_1 + \lambda_2 C_2 \qquad \text{...(7.9)}$$

where J is an $n \times n$ matrix of all ones, λ_1, λ_2 and d are distinct, and the values of λ_1 and λ_2 are given by (7.7). Given the adjacency matrix A of strongly regular graph, we would like to compute the matrices C_1 and C_2. For this purpose, we introduce the following theorem.

Theorem 7.1. Let A be an adjacency matrix of a strongly regular graph $G = (V,E)$ with parameters (n, d, p, q). Let C_1 and C_2 be the matrices satisfying (7.9) which give spectral resolution of A. Then C_1 and C_2, which are square matrices of size n, each contain exactly three distinct values, say x, y, z and x', y', z', whose positions in the matrices C_1 and C_2 are as given below :

(I) $C_1[i, j] = x$

$C_2[i, j] = x'$

for all i, j,

such that $A[i, j] = 1,\ i \neq j$. ...(7.10a)

(II) $C_1[i, j] = y$

$C_2[i, j] = y'$

for all i, j,

such that $A[i, j] = 0,\ i \neq j$. ...(7.10b)

(III) $C_1[i, j] = z$

$C_2[i, j] = z'$

for all i, j,

such that $i = j$. ...(7.10c)

Further, the values of x, y, z and x', y', z' can be expressed solely in terms of the parameters (n, d, p, q) of a given strongly regular graph. They are given by the following expressions.

Let us use the following substitution ;

$$\lambda_1 = (1/2)\{p - q + [(p - q)^2 + 4(d - q)]^{1/2}\} \quad \text{...(7.7a)}$$

$$\lambda_2 = (1/2)\{p - q - [(p - q)^2 + 4(d - q)]^{1/2}\}. \quad \text{...(7.7b)}$$

Then,

$$x = [1 - (1/n)(d - \lambda_2)][\lambda_1 - \lambda_2]^{-1} \quad \text{...(7.11)}$$

$$x' = -[1 - (1/n)(d - \lambda_1)][\lambda_1 - \lambda_2]^{-1}. \quad \text{...(7.12)}$$

$$y = -(1/n)[d - \lambda_2][\lambda_1 - \lambda_2]^{-1} \quad \text{...(7.13)}$$

$$y' = (1/n)[d - \lambda_1][\lambda_1 - \lambda_2]^{-1} \quad \text{...(7.14)}$$

$$z = -[(d/n) + \lambda_2\{1 - (1/n)\}][\lambda_1 - \lambda_2]^{-1}. \quad \text{...(7.15)}$$

$$z' = [(d/n) + \lambda_1\{1 - (1/n)\}][\lambda_1 - \lambda_2]^{-1}. \quad \text{...(7.16)}$$

PROOF : Let us investigate whether we have sufficient information so as to enable us to get the values of six unknowns. Three linear equations can be generated from (7.9). The remaining three can be obtained by recognizing the fact that C_0, C_1 and C_2 are the matrices corresponding to eigenspaces of the *same* real symmetric matrix (namely A). Hence, we have from (6.7), the following equation :

$$C_0 + C_1 + C_2 = \underset{(n \times n)}{I}. \quad \text{...(7.17)}$$

From this, we get the following three equations :

$$(1/n) + x + x' = 0 \quad \text{...(7.18a)}$$

$$(1/n) + y + y' = 0 \quad \text{...(7.18b)}$$

$$(1/n) + z + z' = 1. \quad \text{...(7.18c)}$$

From eqn. (7.9), we get the following set of three equations :

$$(d/n) + \lambda_1 x + \lambda_2 x' = 1 \quad \text{...(7.19a)}$$

$$(d/n) + \lambda_1 y + y_2 y' = 0 \quad \text{...(7.19b)}$$

$$(d/n) + \lambda_1 z + \lambda_2 z' = 0 \quad \text{...(7.19c)}$$

where λ_1, λ_2 are eigenvalues different from d, and are given by (7.7).

It is easy to to see that under the assumption that λ_1, λ_2, and d are distinct, these equations are linearly independent and very easy to solve (consider two equations (7.18a) and (7.19a), solve them for x and x'; similarly other equations and be solved for y

and y', z and z' respectively). It can be verified that the values of x, x', y, y' and z, z' are given by equations (7.11) — (7.16) respectively.

Upto now, we have proved that for strongly regular graph $G = (V,E)$, there exist C_1 and C_2 matrices with structure given in (7.10) containing values given in (711)—(7.16). To note the uniqueness of C_j's, we need to refer to (7.9) which expresses a real symmetric matrix A in terms of its spectral resolution as given by C_j matrices (projection matrices representing the jth eigenspace):

$$A = d\,((1/n)\,J) + \lambda_1 C_1 + \lambda_2 C_2. \qquad \text{...(7.19)}$$

By the uniqueness of C_j matrices for such a representation, the uniqueness of C_1 and C_2 follows. It may be noted that C_j matrices satisfy (6.2) and (6.7) simultaneously, and uniqueness of C_j matrices can be inferred from these two equations. Q.E.D.

The similarity between the above theorem and a property of an adjoint (transpose of matrix of cofactors) of A, where A is an adjacency matrix of strongly regular graph, that was investigated elsewhere[4] is not accidental. In this connection it may be noted that adjoint $(A) = \det(A)\; A^{-1}$. In view of the above theorem, we can now in fact give general expression for function of the matrix of strongly regular graph. We shall not bother about the convergence properties adjacency. The function of a matrix is defined as a series. So, such a matrix function a $f(M)$ can be expressed as[1] :

If

$$M = \sum_{i=1}^{n} \lambda_i \underset{(n\times 1)}{w_i} \underset{(1\times n)}{w_i^T} \qquad \text{...(7.20)}$$

then

$$f(M) = \sum_{i=1}^{n} f(\lambda_i)\, w_i\, w_i^T . \qquad \text{...(7.21)}$$

In terms of C_j matrices, we can express it as :

If

$$M = \sum_{i=1}^{d} \lambda_j C_j \qquad \text{...(7.22)}$$

where $\lambda_1, \lambda_2, \ldots \lambda_d$ are distinct eigenvalues of real symmetric matrix M

then

$$f(M) = \sum_{j=1}^{d} f(\lambda_j)\, C_j. \qquad \text{...(7.23)}$$

Thus, when M is an adjacency matrix of strongly regular graph, computation of $f(M)$ is very much simplified. We have already seen the corresponding C_0, C_1, and C_2 matrices. All that we need to do is to compute $f(\lambda_0 = d)$, $f(\lambda_1)$ and $f(\lambda_2)$ where f is applied to scalars (eigenvalues) and obtain the following sum :

$$f(A) = \sum_{j=0}^{2} f(\lambda_j)\, C_j. \qquad \text{...(7.24)}$$

For the inverse of $A, f(x) = 1/x$ for scalar x. This indicates another method to compute adjoint (A) provided det (A) is known. It may be noted that det (A) is given by the following formula :

$$\det(A) = \lambda_0^{m_0} \lambda_1^{m_1} \lambda_2^{m_2} \qquad \text{...(7.25)}$$

where $\lambda_0 = d$, $m_0 = 1$ and λ_1, λ_2 are given by (7.7) and m_1, m_2 are given by (7.8).

The eigenspace corresponding to the eigenvalue λ_1 is given by the subspace spanned by the rows of C_1 matrix and the eigenspace corresponding to the eigenvalue λ_2 is given by the subspace spanned by the rows of C_2 matrix. Thus, Theorem 7.1 gives us a very neat method for obtaining the eigenspaces of the adjacency matrices of strongly regular graphs. Let us consider the projection matrices C_j $(j = 1, 2)$ corresponding to an eigenvalue λ_j, where $\lambda_j \neq d\,(j = 1, 2)$. The 'structural regularity' in the strongly regular graph is totally represented by any one of the eigenspace corresponding to ony one of the eigenvalues λ_j $(j = 1, 2)$. The following theorem is an easy consequence of this line of thought.

Theorem 7.2= A permutation p is in Aut (G), where G is strongly regular graph, if and only if p is in Aut (C_j), where C_j corresponds to an eigenvalue λ_j which is different from d.

PROOF : From Theorem 6.1 (or, equation 6.8) we can write Aut (G) = Aut (A)

$$= \text{Aut}(C_0) \cap \text{Aut}(C_1) \cap \text{Aut}(C_2). \qquad \text{.. (7.26)}$$

From the structure of C_0, it is clear that Aut (C_0) =Sym (n), a group of all possible permutations on n symbols.

Let us consider a subspace spaned by the rows of C_1. The subspace spanned by all the vectors orthogonal to all the nonzero vectors formed by rows of C_1 has the corresponding C-matrix denoted by $C^{\perp}$. Then from (5.24), we have

$$C_1 + C_1^{T} = I. \qquad \text{...(7.27)}$$

It may be noted from (6.7) that

$$C_0 + C_1 + C_2 = I. \qquad \text{...(7.28)}$$

From (5.36) and (5.37) we have

$$\text{Aut}(C_1) = \text{Aut}(C_1^{\perp}) \qquad \text{...(7.29)}$$

From (7.27) and (7.28) we get

$$C_1^{\perp} = C_0 + C_2. \qquad \text{...(7.30)}$$

Noting that all n^2 entries in C_0 are exactly the same, we can write.

$$\begin{aligned}\text{Aut}(C_1^{\perp}) &= \text{Aut}(C_0) \cap \text{Aut}(C_2)\\ &= \text{Sym}(n) \cap \text{Aut}(C_2)\\ &= \text{Aut}(C_2).\end{aligned} \qquad \text{...(7.31)}$$

From (7.29) and (7.31) we have

$$\text{Aut}(C_1) = \text{Aut}(C_2). \qquad \text{...(7.32)}$$

Substituting this in (7.16) and noting that Aut (C_0) = Sym (n) we get,

$$\begin{aligned}\text{Aut}(G) = \text{Aut}(A) &= \text{Sym}(n) \cap \text{Aut}(C_1) \cap \text{Aut}(C_2)\\ &= \text{Aut}(C_1).\end{aligned} \qquad \text{...(7.33)}$$

Hence the result follows, since Aut (C_1) = Aut (C_2) by (7.32). Q.E.D.

In particular the above theorem implies, for automorphism partitions, the following result :

$$\pi_G = \pi_A = \pi_{C_1} = \pi_{C_2} \qquad \text{...(7.34)}$$

and

$$\pi_{C_0} = u. \qquad \text{...(7.35)}$$

Due to the fact that Aut (C_1) = Aut (G), we may call such subspaces as 'strongly regular sub-spaces'.

Mendelshon[14] has proved an interesting result which says that every finite group can be expressed as a group of automorphisms of a finite strongly regular graph. The natural and interesting consequence of Mendelsohn's result is that every finite group can be viewed as a group of automorphisms of some strongly regular subspace (and hence, as a group of automorphisms of some subspace of $\mathbb{R}^n$). This result is noted in the following theorem.

Theorem 7.3— Let G be a finite group. Then it is isomorphic to a group of automorphisms of some strongly regular subspace.

Sketch of Proof : Let H be a finite group and let A be an adjacency matrix of a strongly regular graph whose group of automorphisms is ismorphic to H. The existence of such a strongly regular graph follows from Mendelsohn[14]. Let $\mathcal{U}$ be a strongly regular subspace associated with A (i. e., an eigenspace of A corresponding to non-d eigenvalue of A, d being the degree of a strongly regular graph). It follows from Theorem 7.2) that the automorphism group associated with the subspace $\mathcal{U}$ is isomorphic to the automorphism group of A. Hence $\mathcal{U}$ is the required strongly regular subspace whose group of automorphisms is isomorphic to the given group H.

Q.E.D.

7.3. *Partially Balanced Designs*

It is well-known that the concept of strongly regular graphs is isomorphic to the concept of association schemes of partially balanced designs[8] with 2-asscociate classes (2-PBIBDS)[3]. Formulae for computing eigenvalues of the matrix NN^T, where N is an incidence matrix of 2-PBIBD, are well-known[11]. In this section, we give a simple method by which we can obtain projection matrices (C_j-matrices) associated with the eigenspaces of the three distinct eigenvalues of NN^T.

For the sake of completeness, we first define 2-association schemes, 2-PBIBDs and then give our main theorem. In this section, we shall denote eigenvalues by θ_0, θ_1, θ_2, since the terms λ_1, λ_2 have special meaning for 2-PBIBDs.

Definition : *2-Association Scheme*

Given v treatments $\{1, 2, \ldots, v\}$, a collection of two relations defined on these treatments satisfying the following conditions is said to from 2-association scheme :

(1) Any two treatments are either first associates or second associates ; the relation of being associates is symmetric.

(2) Every treatments α has n_1 first associates and n_2 second associates.

(3) If treatments α and β are ith associates of each other ($i = 1, 2$), then the number of treatments common to the jth association of α and kth association of β is constant ($= p^i_{jk}$) and is independent of the choice of treatments α and β.

v, n_1, n_2, p^i_{jk} $(i, j, k = 1, 2)$ are parameters of the 2-association scheme. It may be noted that the knowledge of all p^i_{jk}'s is not necessary; for example, given p^1_{12} and p^2_{12}, other are fixed.

Definition : *2-PBIBD*

Given v treatments together with 2-association scheme form a 2-PBIBD if

(1) each treatment occurs at most once in a block (of size k)

(2) each treatment occurs in exactly r blocks

(3) two treatments which are ith associates occur together in λ_i blocks ($i = 1, 2$).

$v, b, r, k, \lambda_1, \lambda_2$ are called the parameters of the 2-PBIBD.

Let N ($v \times b$) be an incidence matrix of 2-PBIBD with parameters v, b, r, k, λ_1, λ_2 whose association scheme has parameters v, n_1, n_2, p_{12}^1, p_{12}^1. It is well-known that the matrix product $M = NN^T$ ($v \times v$) is symmetric and has exactly three distnict eigenvalues, say θ_0, θ_1, θ_2. Out of these, $\theta_0 = rk$ is an eigenvalue of multiplicity one and the corresponding eigenvector is $(1, 1, \ldots, 1)^T$. The other two eigenvalues are given by[11] :

$$\theta_1 = r - (1/2)\,[(\lambda_1 - \lambda_2)(-\nu - \sqrt{\Delta}) + (\lambda_1 + \lambda_2)] \qquad \ldots(7.36)$$

$$\theta_2 = r - (1/2)\,[(\lambda_1 - \lambda_2)(-\nu + \sqrt{\Delta}) + (\lambda_1 + \lambda_2)] \qquad \ldots(7.37)$$

where

$$\nu = p_{12}^2 - p_{12}^1 \qquad \ldots(7.38)$$

$$\eta = p_{12}^2 + p_2^1 \qquad \ldots(7.39)$$

$$\Delta = \nu + 2\eta + 1. \qquad \ldots(7.40)$$

Moreover, their multiplicities, say m_1 and m_2 are given by

$$m_1 = (n_1 + n_2)/2 - \{[(n_1 - n_2) + \nu\,(n_1 + n_2)]/\,(2\sqrt{\Delta})\} \qquad \ldots(7.41)$$

$$m_2 = (n_1 + n_2)/2 + \{[n_1 - n_2) + \nu\,(n_1 + n_2)]/(2\sqrt{\Delta})\}. \qquad \ldots(7.42)$$

Our purpose here is to obtain spectral resolution of $M = NN^T$. From the above discussion, it follows that :

$$M = (rk/v)\,J + \theta_1\,C_1 + \theta_2\,C_2 \qquad \ldots(7.43)$$

where J is $v \times v$ matrix of all ones and θ_1 θ_2, are given by (7.36) and (7.37) respectively.

Theorem 7.4—Let N ($v \times b$) be an incidence matrix of 2-PBIBD with parameters $v, b, r, k, \lambda_1, \lambda_2$ whose association scheme has parameters $v, n_1\, n_2, p_{12}^1\,, p_{12}^2$. Let $M = NN^T$. Let C_1 and C_2 be the matrices satisfying (7.33) which give spectral resolution of M. Then C_1 and C_2, which are square matrices of size v, contain exactly three distinct values, say x, y, z and x', y', z' whose placements in the matrices C_1 and C_2 are as given below :

(I) $C_1\,[i, j] = x$

$C_2\,[i, j] = x'$

for all i, j,

such that $M\,[i, j] = \lambda_1$ ...(7.44a)

(II) $C_1\,[i,j] = y$

$C_2\,[i,j] = y'$

for all i, j,

such that $M\,[i,j] = \lambda_2$. ...(7.44b)

(III) $C_1\,[i,j] = z$

$C_2\,[i,j] = z'$

for all i, j,

such that $M\,[i,j] = r$. ...(7.44c)

Further, the values of x, y, z and x', y', z' can be expressed solely in terms of v, b, r, k, λ_1, λ_2 and n_1, n_2, p^1_{12}, p^2_{12}. Let us denote θ_1 and θ_2 by the expressions given in (7.36) and (7.37) respectively. Then,

$$x = [\lambda_1 - (1/v)\{rk - \theta_2\}]\,[\theta_1 - \theta_2]^{-1} \quad ...(7.45)$$

$$x' = -[\lambda_1 - (1/v)\{rk - \theta_1\}]\,[\theta_1 - \theta_2]^{-1} \quad ...(7.46)$$

$$y = [\lambda_2 - (1/v)\{rk - \theta_2\}]\,[\theta_1 - \theta_2]^{-1} \quad ...(7.47)$$

$$y' = -[\lambda_2 - (1/v)\{rk - \theta_1\}]\,[\theta_1 - \theta_2]^{-1} \quad ...(7.48)$$

$$z = [(r - \theta_2) - (1/v)\{rk - \theta_2\}]\,[\theta_1 - \theta_2]^{-1} \quad ...(7.49)$$

$$z' = -[(r - \theta_1) - (1/v)\{rk - \theta_1\}]\,[\theta_1 - \theta_2]^{-1}. \quad ...(7.50)$$

We omit the proof of this theorem, because it is exactly similar to the proof of Theorem 7.1. We just give the equations whose solutions give the values of x, y, z and x', y', z'. They are :

$$(1/v) + x + x' = 0 \quad ...(7.51a)$$

$$(1/v) + y + y' = 0 \quad ...(7.51b)$$

$$(1/v) + z + z' = 1 \quad ...(7.51c)$$

and

$$(rk)/v + \theta_1 x + \theta_2 x' = \lambda_1 \quad ...(7.52a)$$

$$(rk)/v + \theta_1 y + \theta_2 y' = \lambda_2 \quad ...(7.52b)$$

$$(rk)/v + \theta_1 z + \theta_2 z' = r. \quad ...(7.52c)$$

7.4. *Numerical Examples*

In this part, we give two nonisomorphic strongly regular graphs with parameters $(25, 12, 5, 6)$[6]. They have (the same set of) eigenvalues $\lambda_0 = 12$, $\lambda_1 = 2$, and $\lambda_2 = -3$

with multiplicities $m_0=1$, $m_1=12$ and $m_2=12$ respectively. The eigenspaces corresponding to the eigenvalues λ_1 and λ_2 are strongly regular subspaces (in the sense we introduced in the section 7.2). However, the automorphism partitions of the two eigenspaces associated with eigenvalue $\lambda_1=2$ (or the eigenvalue $\lambda_2=-3$) for these to strongly regular graphs are different. The first strongly regular graph has its automorphism partition as u, thereby giving us an example of subspace (s) with automorphism partition u, (ie., transitive subspaces) as promised in section 5. The second strongly regular graph has the automorphism partition of its vertices $\{1, \ldots, 25\}$ as $\{1\}$, $\{2, 3, \ldots, 13\}$, $\{14, 15, \ldots, 25\}$.

7.4.1. Adjacency listing of the first strongly regular graph :

01:	02	03	04	05	06	07	11	13	16	19	21	25
02:	03	04	05	07	08	12	14	17	20	21	22	
03:	04	05	08	09	13	15	16	18	22	23		
04:	05	09	10	11	14	17	19	23	24			
05:	06	10	12	15	18	20	24	25				
06:	07	08	09	10	11	12	16	18	21	24		
07:	08	09	10	12	13	17	19	22	25			
08:	09	10	13	14	18	20	21	23				
09:	10	14	15	16	19	22	24					
10:	11	15	17	20	23	25						
11:	12	13	14	15	16	17	21	23				
12:	13	14	15	17	18	22	24					
13.	14	15	18	19	23	25						
14:	15	19	20	21	24							
15:	16	20	22	25								
16:	17	18	19	20	21	22						
17:	18	19	20	22	23							
18:	19	20	23	24								
19:	20	24	25									
20:	21	25										
21:	22	23	24	25								
22:	23	24	25									
23:	24	25										
24:	25											
25:												

7.4.2. Adjacency listing of the second strongly regular graph :

01:	02	03	04	05	06	07	08	09	10	11	12	13
02:	03	04	07	08	11	14	15	16	17	21	25	
03:	05	06	09	11	14	15	16	18	20	24		
04:	05	07	08	13	15	17	18	19	20	23		
05:	06	09	13	16	17	18	19	22	25			
06:	07	09	10	14	17	20	21	22	23			
07:	08	10	16	19	20	21	22	24				
08:	09	12	14	18	22	23	24	25				
09:	12	15	19	21	23	24	25					
10:	11	12	13	14	16	17	19	23	24			
11:	12	13	17	18	20	21	24	25				
12:	13	14	15	18	19	21	22					
13:	15	16	20	22	23	25						
14:	15	16	17	18	22	23						
15:	16	19	20	21	23							
16:	19	22	24	25								
17:	18	19	21	23	25							
18:	19	20	22	24								
19:	21	24										
20:	21	22	23	24								
21:	22	25										
22:	25											
23:	24	25										
24:	25											
25:												

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A NOTE ON DISTANCE INCREASING REDUCIBILITY

D. J. POKRASS

Department of Computer Science, Clemson University, Clemson SC 29634-1906

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Given two sets of natural numbers A and B, A is distance increasing (*d. i.*) reducible to B if either $A = B$, or if there exists a recursive function f such that $f(n) > n$ for all n, and $n \in A$ if and only if $f(n) \in B$. We first show that the analog of the Myhill Isomorphism Theorem holds for distance increasing reducibility. Namely, if A and B are each *d. i.* reducible to the other, then they are recursively isomorphic. Our main theorem is that there exist nonrecursive *r. e.* sets A and B which are recursively isomorphic but neither of which is distance increasing reducible to the other.

1. INTRODUCTION

In this paper N represents the set of natural numbers. All functions discussed are defined on a subset of N with range in N. A function f, defined on S, is called partial recursive if there is a Turing machine or program which halts if and only if its input is a member of S, and for all $n \in S$, outputs $f(n)$. Partial recursive functions with domain N are called recursive or computable. A set of natural numbers is recursively enumerable or *r e.* if it is the range of some recursive function. A set is recursive if its characteristic function is recursive.

Recall that a 'reducibility' is a reflexive and transitive binary relation on the subsets of N. Two important reducibilities are m-reducibility and 1-reducibility. Given two sets of natural numbers A and B, A is said to be m-reducible to B, written $A \leqslant_m B$ if there is a recursive function f such that $f(n)$ is in B if and only if n is in A. A is 1-reducible, written $A \leqslant_1 B$, if the function f may be chosen to be also 1–1. (See Odifreddi[3], for an extensive survey on several well-studied reducibilities).

Let us define a recursive function f to be distance increasing if $f(n) > n$ for all n. Let us further define A to be *d.i.*-reducible to B, written $A \leqslant_{di} B$, if either $A = B$ or if $A \leqslant_m B$ via f, for some distance increasing function f. It is clear this is a reducibility.

We mention that the notion of distance increasing resembles the notion of length increasing studied in Berman and Hartmanis[1], but in the latter case functions were defined on strings over an alphabet, and were required to increase the length of the string.

The purpose of this note is to examine some interesting relationships between *d.i.*-reducibility and 1-reducibility. While it is clear that both 1-reducibility and *d. i.* reducibility are special cases of *m*-reducibility, it is also easy to see that neither 1-reducibility nor *d.i.*-reducibility is a consequence of the other. For let $A = \{0, 1\}$ and $B = \{2\}$. Then $A \leqslant_{di} B$, but $\neg A \leqslant_1 B$ since A has larger cardinality than B. Also we have $B \leqslant_1 A$, but $7 \; B \leqslant_{di} A$ since a member of B is greater than all members of A.

Although a distance increasing function is not necessarily 1–1, it comes "close" to being 1–1 in the sense that $f^{-1}(\{n\})$ is always a finite set. In the next section we observe another similarity between 1-reducibility and *d.i.*-reducibiliyt. In the last section, we illustrate a fundamental difference.

2. Myhill's Isomorphism Theorem

An important theorem of Myhill[2] states : If $A \leqslant_1 B$ and $B \leqslant_1 A$ then there is 1–1 recursive function f which maps N onto N and A onto B. In this case A and B are said to be recursively isomorphic and we write $A \equiv B$. This theorem is often said to be an analog to the well-known Schroeder-Bernstein Theorem[5] of set theory which states that if A and B are (arbitrary) sets such that each can be mapped with 1-1 functions into a subset of the other, then there is a one-to-one correspondence between A and B. An analog of this theorem also holds for *d.i.*-reducibility :

Proposition—Let A and B be sets of natural numbers, and assume $A \leqslant_{di} B$ and $B \leqslant_{di} A$. Then there is 1–1 recursive function which maps N onto N and A onto B.

Proof : The proposition is obvious in the case when $A = B$, so assume $A \neq B$. Then there are distance increasing recursive functions f and g such that $A \leqslant_{di} B$ via f and $B \leqslant_{di} A$ via g. By Myhill's theorem, it suffices to show $A \leqslant_1 B$ and $B \leqslant_1 A$. To show $A \leqslant_1 B$ we construct a 1–1 recursive function h such that $x \in A$ if and only if $h(x) \in B$. The recursive function h is defined inductively. First let $h(0) = f(0)$, and now assume that each $h(i)$ has been defined for $i < k$. For $i=0, 1, 2...$ define $\{n_i\}$ to be the sequence

$$f(k), f(g(f(k)), f(g(f(g(f(k)))) \ldots$$

We note that all members of the sequence are distinct since the sequence is strictly increasing. Define $h(k)$ to be the first term in the sequence not equal to any of the previously defined values $h(i)$, for $i < k$. It is clear that h is both recursive and 1–1. It remains to show that $x \in A$ if and only if $h(x) \in B$. However by our assumption of f and g we know

$$x \in A \text{ iff } f(x) \in B \text{ iff } g(f(x)) \in A \text{ iff } f(g(f(x))) \in B \ldots$$

and so $A \leqslant_1 B$. By a similar argument we can show $B \leqslant_1 A$.

3. Recursively Enumerable Sets

In this section we construct nonrecursive r.e. sets A and B such that $A \equiv B$ but neither A nor B is *d.i.*-reducible to the other. We assume $\{\phi_e\}$, $e = 0, 1, 2,\ldots$ is a

standard numbering of the partial recursive functions. We write $\phi_{e,s}(x) \downarrow$ to mean that the eth Turing machine, which defines ϕ_e, halts for input x in less than s steps to some output value y, where e, x and y are each less than s.

Lemma 1—If A is a recursive set and both A and $\sim A$ are infinite, then there is a distance increasing function f such that $A \leqslant_{di} A$ via f.

PROOF : By the assumptions about A there are recursive functions g_1 and g_2 such that

$$g_1(0), g_1(1)\, g_1(2), \ldots$$

and

$$g(0), g_2(1), g_2(2), \ldots$$

are, respectively, strictly, increasing enumerations of the members of A and $\sim A$. We may take $f(n)$ to be $g_1(n+1)$ if $n \in A$, and $g_2(n+1)$ otherwise. This is a distance increasing function since $f(n) = g_i(n+1) > n$.

Theorem—There exists a nonrecursive *r.e.* set A such that for any distance increasing recursive function f, A is not reducible to itself via f.

Construction—We construct A in stages. At each stage we enumerate at most one new element into A. Hence there will be a recursive sequence which enumerates the set A, and A will be *r.e.* The construction makes use of the well-known priority method[2,4]. For each partial recursive function ϕ_e, $e = 0, 1, 2, \ldots$ we must satisfy the requirement

$$R_e : \neg\, A \leqslant_{di} A \text{ via } \phi_e.$$

That is, we must prevent A from being *d.i.*-reducible to itself via ϕ_e. At the same time we will build A so that both it and its complement are infinite. Lemma 1 will then insure that A is nonrecursive.

During the construction, at each new stage s, a restraint function $r(e, s)$ is updated for each value e. At stage 0, $r(e, 0)$ is initialized to -1 for all e. Once $r(e, s)$ becomes nonnegative, requirement R_e will be maintained as long as new elements are always greater than $r(e, s)$. At each stages $s + 1$ we try not to injure requirements R_e and hence try to keep incoming elements greater than $r(e, s)$. This is not always possible, and like typical "priority arguments" is given to lower numbered requirements.

During the construction which follows, odd stages will be used to insure A and $\sim A$ are infinite, and even stages will be used to meet the requirements R_e. Elements for A are always drawn from two infinite recursive sets W and W'. We assume

$$W' = \{n_0 < n_1 < n_2 < \ldots\}$$

and that W and W' are disjoint. The set A_i designates the elements enumerated into A at the end of stage i.

Stage 0 : Initialize $A_0 = \{\}$ and $r(e, 0) = -1$ for all e.

Stage $2s + 1$: Euumerate into A the least $n_i \in W' - A_{2s}$ such that i is odd and $n_i > r(e, 2s)$ for all e. Set $r(e, 2s + 1) = r(e, 2s)$ for all e.

Stage $2s + 2$: Agree that R_e requires attention if

(a) $r(e, 2s + 1) = -1$, and

(b) There is some $x_0 \in W - A_{2s+1}$ such that $x_0 < \phi_{e,s}(x_0)\downarrow$ and $x_0 > r(e', 2s+1)$ for all $e' < e$.

If no requirement at this stage requires attention then define $A_{2s+2} = A_{2s+1}$ and $r(e', 2s + 2) = r(e', 2s + 1)$ for all e'. Otherwise let e be smallest such that R_e requires attention, and let x_0 be a number satisfying (b). Then R_e receives attention and we do the following :

Case 1—$\phi_{e,s}(x_0) \in A_{2s+1}$. Define $r(e, 2s + 2) = x_0 + 1$, and for all other e', let $r(e', 2s + 2) = r(e', 2s + 1)$. Hence we try to keep x_0 from entering A.

Case 2—otherwise. Enumerate x_0 into A. Define $r(e, 2s + 2) = \phi_{e,s}(x) + 1$ Hence we try to keep $\phi_e(x_0)$ from entering A. For $e' < e$ keep $r(e', 2s + 2)$ the same as $r(e', 2s + 1)$ but for $e' > e$ reset $r(e', 2s + 2) = -1$.

The following lemmas complete the proof of the theorem.

Lemma 2—Each requirement R_e receives attention at most 2^e times.

PROOF : Note $r(e, s) = -1$ and $r(e, s + 1) \geqslant 0$ if and only if R_e receives attention at stage $s + 1$. Furthermore, $r(e, s) \geqslant 0$ and $r(e, s + 1) = -1$ if and only if R_{e_1}, for some $e_1 < e$, received attention at stage $s + 1$. It follows that R_0 can receive attention at most once. Assume by induction that the lemma is true for all $e_1 < e$, and let $s_1, s_2, \ldots$ be the stages at which R_e, receives attention. Then for each $i > 1$ there is some $e_i < e$ for which R_{e_i} receives attention between stages s_{i-1} and s_i. By our induction assumption, after R_e receives attention for the first time, it can only receive attention $2^e - 1$ more times. This completes the proof.

Lemma 3—Assume $r(e, s) \geqslant 0$ and for each $e' < e$, $R_{e'}$ never receives attention after stage s. Then A satisfies requirement R_e.

PROOF : Let s_0 be the last stage at which R_e receives attention. By our comments above, $s_0 \leqslant s$, and s_0 is the last stage at which any $R_{e'}$, $e' \leqslant e$, receives attention. At the end of stage s_0, there is a number $x_0 < \phi_e(x_0)$ such that exactly one member of the pair $(x_0, \phi_e(x_0))$ has been placed in A. Requirement R_e will be satisfied if the other member of the pair does not ever enter A. However, at the end of stage s_0, $r(e, s_0)$ is defined to be greater than the member in the pair not enumerated into A. Since at every stage following s_0, any new element enumerated into A is always greater than $r(e, s_0)$, the lemma follows.

Lemma 4—A and $\sim A$ are infinite and all requirements R_e are satisfied.

PROOF : A is infinite since at each odd numbered stage we enumerate an element from W' $\sim A$ is infinite since we exclude from A all n_i for even i. Let e be fixed, and we now show R_e is satisfied. We may assume for all x $\phi_e(x)$ is defined and $\phi_e(x) > x$. By Lemma 2 we may choose s so that no requirement $R_{e''}$, for $e < e$, ever receives attention at stage s or later. If $r(e, s) \geqslant 0$ then R_e will be satisfied by Lemma 3. Assume, then, $r(e, s) = -1$. Choose $x_0 \in W - A$, such that $x_0 \geqslant r(e', s)$ for all $e' < e$. Now let $s' \geqslant s$ be minimal such that $\phi_{e,s'}(x_0)\downarrow$. Then R_e requires and receives attention no later than stage $2s' + 2$, and so $r(e, 2s' + 2)$ becomes non-negative. Again, by Lemma 3, R_e will be satisfied.

We now may prove the result mentioned in the beginning of this section.

Corollary—There exist nonrecursive *r.e.* sets A and B such that $A \equiv B$ but neither $A \leqslant_{di} B$, nor $B \leqslant_{di} A$.

PROOF : Let A be the set constructed in the previous theorem. Let a_0 be the smallest number in A, and a_0' the smallest number not in A. Let $B = A \cup \{a_0'\} - \{a_0\}$. Then A and B are easily seen to be recursively isomorphic via the function which maps a_0' to a_0, a_0 to a_0', and leaves all other numbers fixed. However suppose $A \leqslant_{di} B$ via the distance increasing function $f(x)$. Then let C be the finite set of x for which $f(x) = a_0$, and C' the finite set of x for which $f(x) = a_0'$. Let a_1 be the smallest element in A, greater than a_0', and let a_1' be the smallest element not in A, but greater than a_0. Then let f_1 be the function which maps x to a_1' for $x \in C$, x to a_1 for $x \in C'$, and all other x to $f(x)$. It is easy to see that f_1 is distance increasing and that A is *d.i.*-reducible to itself via f_1, a contradiction. A similar argument may be used to show $\neg B \leqslant_{di} A$.

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FIXED POINT AND COINCIDENCE POINT THEOREMS

S. V. R. NAIDU

Department of Applied Mathematics, A.U.P.G. Extension Centre Nuzvid 521201

AND

K. P. R. RAO

D. A. R. College, Nuzvid 521201

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Certain common fixed point theorems for pairs of selfmaps on a pseudo-compact topological space and pairs as well as families of selfmaps on a compact metric space are obtained. Also certain sufficient conditions for the existence of coincidence points for at least one pair of four maps (two of which are multi-functions) on a compact metric space are discussed.

The paper is divided into two sections. In section 1, we obtain few fixed point theorems for a pair of selfmaps on a pseudo-compact topological space when the maps together satisfy certain generalized contractive conditions with reference to a metric-type function on the space. Incidentally, we observe that the theorems of Pathak[4] are false and suggest some suitable modifications for them. We also obtain a couple of fixed point theorems for pairs as well as families of selfmaps on a compact metric space.

In section 2, we consider two pairs of maps on a compact metric space, each pair consisting of selfmap on the space and a multi-function, and discuss the existence of a coincidence point for one pair when all the four maps together satisfy a certain generalized contractive condition.

In each section we provide a number of examples to throw light on many of the condition stipulated in the theorems.

Throughout this paper, R denotes the set of all real numbers R^+ the set of all nonnegative real numbers. For a subset A of a metric space, $\delta(A)$ denotes the diameter of A.

SECTION 1

We begin with the following known definitions.

Definition 1.1—A topological space is said to be pseudo-compact if every real-valued continuous function on it is bounded.

We note that every real-valued continuous function on a pseudo-compact topological space attains its bounds.

Definition 1.2—A pair of selfmaps f, P on a metric space (X, d) are said to be weakly commutative if $d(fPx, Pfx) \leqslant d(Px, fx)$ for all x in X.

Throughout this section, unless otherwise stated, X stands for a pseudo-compact topological space and μ denotes a nonnegative real-valued function on $X \times X$ such that $\mu(x, x) = 0$ for all x in X and $\mu(x, y) \leqslant \mu(x, z) + \mu(z, y)$ for all x, y, z in X.

Theorem 1.1—Let f and P be selfmaps on X such that

(1) $f(X) \subseteq P(X)$,

(2) the function ϕ defined on X by $\phi(x) = \mu(fx, Px)$ is continuous on X and

(3)
$$\begin{aligned}\mu(fx, fy) < \max\{&\beta_1(x, y)\,\mu(Px, Py) + \beta_2(x, y)\,\mu(fx, Px)\\ &+ \beta_3(x, y)\,\mu(fy, Py) + \beta_4(x, y)\,\mu(fx, Py)\\ &+ \beta_5(x, y)\,\mu(Px, fy), \gamma_1(x, y)\,\mu(Px, Py),\\ &\gamma_2(x, y)\,\mu(fx, Px), \gamma_3(x, y)\,\mu(fy, Py),\\ &\gamma_4(x, y)\,\mu(fx, Py), \gamma_5(x, y)\,\mu(Px, fy),\\ &\tfrac{1}{2}[\mu(fx, Py) + \mu(Px, fy)]\}\end{aligned} \qquad \ldots(1.1)$$

whenever x, y in X are such that $fx \neq fy$ and $Px \neq Py$, where β_i's are real-valued and γ_i's are nonnegative real-valued functions on $X \times X$. If

(4) $\beta_1(x, y) + \beta_2(x, y) + \beta_3(x, y) + 2\beta_4(x, y) \leqslant 1$,

(5) $\beta_2(x, y) + \beta_4(x, y) < 1$,

(6) $\beta_4(x, y) \geqslant 0$

(7) $\gamma_i(x, y) \leqslant 1$ $(i = 1, 2, 3)$ and

(8) $\gamma_4(x, y) \leqslant \frac{1}{2}$,

whenever $Px = fy, fx \neq fy$ and $Px \neq Py$, then f and P have a coincidence point.

If f and P have a coincidence point z such that

(9) $fPz = Pfz$

(10) $\beta_1(Pz, z) + \beta_4(Pz, z) + \beta_5(Pz, z) \leqslant 1$ and

(11) $\gamma_i(Pz, z) \leqslant 1$ $(i = 1, 4, 5)$,

then Pz is a common fixed point of f and P.

If $B_1(x, y) + \beta_4(x, y) + \beta_5(x, y) \leqslant 1$ and $\gamma_i(x, y) \leqslant 1$ $(i = 1, 4, 5)$ for all coincidence points x, y of f and P, then f and P have at most one common fixed point.

PROOF : Since X is pseudo-compact, from (2) it follows that there exists a v in X such that $\phi(v) = \min \{\phi(x)/x \in X\}$. From (1), there exists a w in X such that $fv = Pw$. Suppose that (4) to (8) hold whenever $Px = fy, fx \neq fy$ and $Px \neq Py$. If possible, suppose now that $fv \neq fw$ and $Pv \neq Pw$. Since μ satisfies triangle inequality and $fv = Pw$ we have

$$\mu(fw, Pv) \leqslant \phi(w) + \phi(v).$$

Taking $x = w$ and $y = v$ in inequality (1.1), and making use of (6), (7) and (8) with $x = w$ and $y = v$, the above inequality, the equation $fv = Pw$, the nonnegativity of μ and the fact that μ vanishes when its arguments are equal, we get

$$\phi(w) < \max \{[\beta_1(w, v) + \beta_3(w, v) + \beta_4(w, v)]\, \phi(v) + [\beta_2(w, v) + \beta_4(w, v)]\, \phi(w), \phi(v), \phi\, w)\}.$$

Since (4) and (5) are true for $x = w$ and $y = v$, ϕ is nonnegative and $\phi(v) \leqslant \phi(w)$, from the above inequality we get $\phi(w) < \phi(w)$ which is a contradiction.

Hence either $fv = fw$ or $Pv = Pw$.

Since $fv = Pw$ it follows that either v or w is a coincidence point of f and P.

Suppose that z is a coincidence point of f and P such that (9), (10) and (11) hold.

Since $fz = Pz$ and $fPz = Pfz$ we have $fPz = P^2z$.

If possible, suppose $P^2z \neq Pz$. Then $fPz \neq fz$.

Taking $x = Pz$ and $y = z$ in inequality (1.1) and making use of (10), (11), the equations $fz = Pz, fPz = P^2z$, the nonnegativity of μ and the fact that μ vanishes when its arguments are equal, we get $\mu(P^2z, Pz) < \mu(P^2z, Pz)$ which is a contradiction.

Hence $P^2z = Pz$.

Hence Pz is a common fixed point of f and P.

The rest of the theorem is evident from inequality (1.1).

Remark (1.1) : Example 1.1 [Example 1.2] shows that in Theorem (1.1) one cannot conclude that the P-image of a coincidence point z of f and P is a common fixed point of f and P if condition (9) (condition (10)) is dropped, even if (X, μ) is a compact metric space; f, P are continuous on X; the β_i's are all constants; the γ_i's are all zeros; and inequality (1.1) holds whenever $Px \neq Py$.

Example 1.1—Let $X = \{0, 1\}$ with the usual metric. Define f, P from X into X as $f0 = f1 = 0$, $P0 = 1$ and $P1 = 0$. Then for any positive constant β_1, we have

$$|fx - fy| < \beta_1 |Px - Py|$$

whenever $Px \neq Py$.

Example 1.2—Let X {0, 1} with the usual metric and define $f, P : X \to X$ as $f0 = 1 = P0, f1 = 0 = P1$. Then

$$| fx - fy | < | Px - Py | + \beta_5 | Px - fy |$$

whenever $Px \neq Py$ (i.e., $x \neq y$), where β_5 is a positive constant.

Remark 1.2 : Example 1.3 shows that in Theorem 1.1 one cannot ensure the uniqueness of common fixed point for f and P if the condition : "$\beta_1(x, y) + \beta_4(x, y) + \beta_5(x, y) \leqslant 1$ for all coincidence points x, y of f and P" is dropped, even if (X, μ) is a compact metric space, f and P are commutative continuous maps, the γ_i's are all zeros and inequality (1.1) holds whenever $Px \neq Py$.

Example 1.3—Let $X = \{0, 1\}$ with the usual metric and f, P be identity maps on X. Then

$$| fx - fy | < \beta_1(x, y) | Px - Py | + \beta_5(x, y) | Px - fy |$$

whenever $x \neq y$, where

$$\beta_1(x,y) = \begin{cases} 1 & \text{if } x \neq y, \\ 0 & \text{otherwise,} \end{cases} \quad \text{and } \beta_5(x, y) = \begin{cases} \alpha & \text{if } x \neq y, \\ 0 & \text{otherwise,} \end{cases}$$

α being a positive constant.

Corollary 1.1—Suppose that $\mu(x, y) > 0$ for all distinct x, y in X. Let f and P be selfmaps on X such that

(1) $f(X) \subseteq P(X)$,

(2) the function ϕ defined on X as $\phi(x) = \mu(fx, Px)$ is continuous on X and

(3) $$\mu(fx, fy) < \max \Big\{a_1(x, y)\,\mu(Px, Py) + a_2(x, y)\,\mu(fx, Px) + a_3(x, y)\,\mu(fy, Py) + a_4(x, y)\,\mu(fx, Py) + a_5(x, y)\,\mu(Px, fy) + a_6(x, y)\,\frac{\mu(fx, Px)\,\mu(fy, Py)}{\mu(Px, Py)} + a_7(x, y)\,\frac{\mu(fx, Py)\,\mu(Px, fy)}{\mu(Px, Py)},$$
$$b_1(x, y)\,\mu(Px, Py),\ b_2(x, y)\,\mu(fx, Px),$$
$$b_3(x, y)\,\mu(fy, Py),\ b_4(x, y)\,\mu(fx, Py),$$
$$b_5(x, y)\,\mu(Px, fy),\ b_6(x, y)\,\frac{\mu(fx, Px)\,\mu(fy, Py)}{\mu(Px, Py)},$$
$$b_7(x, y)\,\frac{\mu(fx, Py)\,\mu(Px, fy)}{\mu(Px, Py)},$$
$$\tfrac{1}{2}[\mu(fx, Py) + \mu(Px, fy)]\Big\} \qquad \text{...(1.2)}$$

whenever x, y in X are such that $fx \neq fy$ and $Px \neq Py$, where a_i's are real-valued and b_i's are nonnegative real-valued functions on $X \times X$ such that

(i) $a_1(x, y) + a_2(x, y) + a_3(x, y) + 2a_4(x, y) + a_6(x, y) \leqslant 1$,

(ii) $a_2(x, y) + a_4(x, y) + a_6(x, y) < 1$,

(iii) $a_4(x, y) \geqslant 0$,

(iv) $b_i(x, y) \leqslant 1$ $(i = 1, 2, 3, 6)$ and

(v) $b_4(x, y) \leqslant \frac{1}{2}$

whenever $Px = fy$, $fx \neq fy$ and $Px \neq Py$, then f and P have a coincidence point.

If f and P have a coincidence point z such that $fPz = Pfz$, $a_1(Pz, z) + a_4(Pz, z) + a_5(Pz, z) + a_7(Pz, z) \leqslant 1$ and $b_i(Pz, z) \leqslant 1$ $(i = 1, 4, 5, 7)$, then Pz is a common fixed point of f and P.

If $a_1(x, y) + a_4(x, y) + a_5(x, y) + a_7(x, y) \leqslant 1$ and $b_i(x, y) \leqslant 1$ $(i = 1, 4, 5, 7)$ for all conincidence points x, y of f and P then f and P have at most one common fixed point.

Proof : Define $\beta_1, \ldots, \beta_5$ and $\gamma_1, \ldots, \gamma_5$ from $X \times X$ to R as $\beta_i(x, y) = a_i(x, y)$, $\gamma_i(x, y) = b_i(x, y)$ for $i = 1, 3, 4$,

$$\beta_2(x, y) = \begin{cases} a_2(x, y) + a_6(x, y) \dfrac{\mu(fy, Py)}{\mu(Px, Py)} & \text{if } Px \neq Py, \\ a_2(x, y) + a_6(x, y) & \text{otherwise,} \end{cases}$$

$$\beta_5(x, y) = \begin{cases} a_5(x, y) + a_7(x, y) \dfrac{\mu(fx, Py)}{\mu(Px, Py)} & \text{if } Px \neq Py, \\ a_5(x, y) + a_7(x, y) & \text{otherwise,} \end{cases}$$

$$\gamma_2(x, y) = \begin{cases} \max \left\{ b_2(x, y), b_6(x, y) \dfrac{\mu(fy, Py)}{\mu(Px, Py)} \right\} & \text{if } Px \neq Py, \\ \max \{ b_2(x, y), b_6(x, y) \} & \text{otherwise,} \end{cases}$$

and

$$\gamma_5(x, y) = \begin{cases} \max \left\{ b_5(x, y), b_7(x, y) \dfrac{\mu(fx, Py)}{\mu(Px, Py)} \right\} & \text{if } Px \neq Py, \\ \max \{ b_5(x, y), b_7(x, y) \} & \text{otherwise.} \end{cases}$$

Then it can be verified that the conditions on a_i's and b_i's imply the corresponding conditions on β_i's and γ_i's of Theorem 1.1 and that for $Px \neq Py$ inequality (1.2) is equivalent to inequality (1.1). Hence corollary follows.

Remark 1.3 : Corollary 1.1 is a generalization of Theorem 1 of Jain and Dixit[3]. The versions of Theorem 1.1 and Corollary 1.1 corresponding to Theorem 2 of Jain and Dixit[3] [i.e., the ones obtained by dropping the terms involving $\mu(fx, Py)$ and

$\mu(Px, fy)$ from inequalities (1.1) and (1.2)] are valid even in the absence of triangular inequality of μ.

Corollary 1.2—Suppose that $\mu(x, y) > 0$ for all distinct x, y in X. Let f, g be continuous selfmaps on X such that

$$\mu(fgx, fy) < \alpha_1 \mu(gx, y) + \alpha_2 \mu(fgx, gx) + \alpha_3 \mu(fy, y) + \alpha_4 \mu(fgx, y) + \alpha_5 \mu(gx, fy) + \alpha_6 \frac{\mu(fgx, gx)\,\mu(fy, y)}{\mu(gx, y)} + \alpha_7 \frac{\mu(fgx, y)\,\mu(gx, fy)}{\mu(gx, y)} \quad ...(1.3)$$

for all x in X and y in $g(X)$ such that $fgx \neq fy$, where the α_i's are constants such that $\alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_6 \leqslant 1$, $\alpha_2 + \alpha_4 + \alpha_6 < 1$ and $\alpha_4 \geqslant 0$. Then f has a fixed point. If, in addition, $\alpha_1 + \alpha_4 + \alpha_5 + \alpha_7 \leqslant 1$ then f has a unique fixed point in $g(X)$. If, further, $fg = gf$ then f and g have a unique common fixed point in X.

PROOF: Since X is pseudo-compact and g is a continuous selfmap on X, it is clear that $g(X)$ is also pseudo-compact. It is easy to see that the hypothesis of Corollary 1.1 holds when we take $g(X)$ in the place of X; the identity map on $g(X)$ in the place of P; $a_i(x, y) = \alpha_i$ and $b_i(x, y) = 0$ $(i = 1, 2, ..., 7)$. Hence from Corollary 1.1 it follows that f has a fixed point in $g(X)$ and that it is unique in $g(X)$ when $\alpha_1 + \alpha_4 + \alpha_5 + \alpha_7 \leqslant 1$. Suppose now that $fg = gf$ and v is the unique fixed point of f in $g(X)$. Then $fgv = gfv = gv$ so that gv is a fixed point of f in $g(X)$ and hence $gv = v$. Since any common fixed point of f and g belongs to $g(X)$, it follows that f and g have a unique common fixed point in X.

Remark 1.4: Example 1.4 shows that Theorem 1 of Pathak[4] is false. Corollary 1.2 is a possible modification of it. Example 1.1 also shows that Theorems 2 and 3 of Pathak[4] are also false. In all these theorems Pathak tacitly assumes that $\mu(x, y) > 0$ for $x \neq y$. With this additional stipulation, Theorems 2 and 3 of Pathak[4] remain valid if the inequalities in them which are assumed to hold whenever $x \neq y$ and $Tx \neq y$ are constrained to hold whenever $Tx \neq y$.

Example 1.4—Let $X = \{0, 1\}$ with the usual metric μ. Define $f, g : X \to X$ as $f0 = 0, f1 = 1, g0 = 1$ and $g1 = 0$. Clearly f, g are commutative, continuous selfmaps on X and g has no fixed point. There are no x, y in X for which $x \neq y$ and $gx \neq y$ hence inequality (1.3) is vacuously satisfied for $x \neq y$ and $gx \neq y$, and for any constants $\alpha_1, ..., \alpha_7$.

Corollary 1.3—Let (X, d) be a compact metric space and f, P be weakly commuting continuous selfmaps on X such that $f(X) \subseteq P(X)$ and

$$d(fx, fy) < \max\{\beta_1 d(Px, Py) + \beta_2 d(fx, Px) + \beta_3 d(fy, Py) + \beta_4 d(fx, Py) + \beta_5 d(Px, fy), d(Px, Py),$$

$$d(fx, Px), d(fy, Py),$$
$$\tfrac{1}{2}[d(fx, Py) + d(Px, fy)], d(Px, fy)\} \qquad \ldots(1.4)$$

whenever x, y in X are such that $Px \neq Py$, where β_i's are real constants such that $\beta_4 \geqslant 0$, $\beta_2 + \beta_4 < 1$, $\beta_1 + \beta_2 + \beta_3 + 2\beta_4 \leqslant 1$ and $\beta_1 + \beta_4 + \beta_5 \leqslant 1$. Then f and P have a unique common fixed point.

Remark 1.5 : Theorem 1.1 remains valid if ϕ is defined on X as $\phi(x) = \mu(Px, fx)$, the factors $\mu(fx, Px)$, $\mu(fy, Py)$, $\mu(fx, Py)$, $\mu(Px, fy)$ in the expressions on the right-hand side of inequality (1.1) are replaced by $\mu(Py, fy)$, $\mu(Px, fx)$, $\mu(Px, fy)$ $\mu(fx, Py)$ respectively, and the equation $Px = fy$ (following (8)) is replaced by $fx = Py$. Similar remarks hold for Corollaries 1.1, 1.2 and 1.3 also.

Remark 1 6 : Example 1.5 shows that in Corollary 1.3 one cannot prove the existence of a common fixed point for f and P (even when P is the identity map and the B_i's are nonnegative) if the average of $d(fx, fy)$ and $d(Px, fy)$ on the right-hand side of inequality (1.4) is replaced by $d(fx, Py)$.

Example 1.5—Let $X = \{z \in \mathbb{C} : |z| = 1\}$ with the usual metric. Define $f : X \to X$ as $fz = iz$.

Then f is continuous on X and

$$|fz_1 - fz_2| < \max\{\tfrac{1}{2}|fz_1 - z_1| + \tfrac{1}{2}|fz_2 - z_2| + |z_1 - fz_2|,$$
$$|fz_1 - z_2|\}$$

for all z_1, z_2 in X.

Clearly f has no fixed point in X.

Corollary 1.3 remains valid when inequality (1.4) is modified as specified in Remark 1.6 provided that P is one-to-one and β_i's are nonnegative constants with $\beta_1 + \ldots + \beta_5 \leqslant 1$. In fact, we have the following

Theorem 1.2—Let (X, d) be a compact metric space and f, P be weakly commuting continuous selfmaps on X such that there is a sequence $\{x_n\}$ in X such that $fx_n = Px_{n+1}$ $(n = 0, 1, 2, \ldots)$, and

$$d(fx, fy) < \max\{d(Px, Py), d(fx, Px), d(fy, Py), d(fx, Py),$$
$$d(Px, fy)\} \qquad \ldots(1.5)$$

whenever x, y in X are such that $fx \neq fy$. Then f and P have a uniqne common fixed point.

Remark 1.7 : Example 1.6 shows that the condition regarding the existence of $\{x_n\}$ cannot be dropped from Theorem 1.2.

Example 1.6—Let $X = [0, 1]$ with the usual metric. Define $f, P : X \to X$ as $fx = x/2$ and $Px = \frac{1}{2}(1 + x)$.

Then f, P are continuous on X, P is one-to-one on X,

$$| fPx - Pfx | = \tfrac{1}{4} < \tfrac{1}{2} = | fx - Px |$$

for all x in X and

$$| fx - fy | < \max \{ | fx - Px | , | fy - Py | , | fx - Py | , Px - fy | \}$$

for all x, y in X.

But f and P have no common fixed point.

Remark 1.8 : Example 1.7 shows that the weak commutativity of f and P cannot be dropped from Theorem 1.2 even if inequality (1.5) holds for all distinct x, y in X.

Example 1.7—Let $X = [0, 1]$ with the usual metric. Define $f, P : X \to X$ as $fx = 0$ and $Px = 1 - x$.

Then f, P are continuous on X, P is one-to-one on X, $f(X) \subset P(X)$ and

$$| fx - fy | < | Px - Py |$$

for all distinct x, y in X.

But f and P have no common fixed point.

Remark 1.9 : Inequality (1.5) is valid whenever $fx \neq fy$ iff it is valid whenever its right-hand side is positive. In view of this observation, Theorem 1.2 is an improvement over Theorem 5 of Fisher[1]. In the latter, it is assumed that $fP = Pf$ and $f(X) \subseteq P(X)$.

Theorem 1.2 can be easily deduced from the following

Theorem 1.3—Let (X, d) be a compact metric space and f, P be weakly commuting continuous selfmaps on X such that

(1) there is a sequences $\{x_n\}$ in X such that $fx_n = Px_{n+1}$ $(n = 0, 1, 2, \ldots)$ and

(2) $\delta(f(A)) = 0$ whenever A is a nonempty closed subset of X such that $f(A) = P(A)$.

Then f and P have a unique common fixed point in X.

Proof : Let S denote the set of all subsequential limits of $\{x_n\}$. Then S is a nonempty compact subset of X and $f(S) = P(S)$. Hence $f(S) = P(S) = \{z\}$ for some z in X. For any u is S we have $fu = Pu = z$ so that from the weak commutativity of f and P we get $fz = Pz$. For $T = \{u\} \cup \{z\}$ we have $f(T) = P(T)$ so that $fu = fz$. Hence z is a common fixed point of f and P. Uniqueness of common fixed point is evident from (2).

Remark 1.10 : Example 1.8 shows that Theorem 1.3 is more general than Theorem 1.2.

Example 1.8—Let $X = [0, 1]$ with the usual metric. Define $f, P : X \to X$ as $fx = x$ and $Px = 0$ for all x in X. For $x \neq 0$ and $y = 0$ inequality (1.5) is not valid.

Theorem 1.3 can be improved in many aspects if the weak commutativity of the maps in it is replaced by the more restrictive condition commutativity. In fact, we have the following.

Theorem 1.4—Let (X, d) be a compact metric space and f, g be commutative selfmaps on X. Let $\mathscr{F}$ be a nonempty family of selfmaps on X. Suppose that for some positive integer p, $(fg)^p$ is continuous on X and commutes with every member of $\mathscr{F}$. Suppose also that $\delta(A) = 0$ whenever A is a minimal nonempty closed subset of X such that $f(A) = g(A) = A$ and $P(A) \subseteq A$ for all P in $\mathscr{F}$. Then the family $\mathscr{F} \cup \{f\} \cup \{g\}$ has a common fixed point in X.

PROOF: Let $\tau = \{A/A$ is a nonempty, closed subset of $X, f(A) = g(A) = A$ and $P(A) \subseteq A \ \forall \ P$ in $\mathscr{F}\}$. Then τ is nonempty since $\bigcap\limits_{n=1}^{\infty} (fg)^{pn} X$ is a member of τ. For $A, B \in \tau$ define $A \leqslant B$ iff B is a subset of A.

Then $\leqslant$ is a partial order on τ.

If τ' is a nonempty chain in τ, then, for $A^1 = \bigcap\limits_{A \in \tau'} A$, it is easy to see that $\bigcap\limits_{n=1}^{\infty} (fg)^{pn} A^1$ is an upper bound of τ' in τ. Hence, by Zorn's lemma, τ has a maximal element $\tilde{A}$. From the hypothesis, $\tilde{A} = \{z\}$ for some z in X. Hence z is a common fixed point of $\mathscr{F} \cup \{f\} \cup \{g\}$.

Corollary 1.4—Let (X, d) be a compact metric space and f, g be commutative selfmaps on X. Let $\mathscr{F}$ be a nonempty family of selfmaps on X. Suppose that for some positive integer p, $(fg)^p$ is continuous on X and commutes with every member of $\mathscr{F}$. Suppose also that

$$d(f^r x, g^s y) < \delta(\mathscr{O}(x) \cup \mathscr{O}(y))$$

whenever $f^r x \neq g^s y$, where r, s are fixed positive integers and $\mathscr{O}(x) = \{Px \mid P \in \tau\}$, τ being the semigroup of selfmaps on X generated by $\mathscr{F} \cup \{f\} \cup \{g\} \cup \{I\}$ (I being the identity map on X). Then the family $\mathscr{F} \cup \{f\} \cup \{g\}$ has a unique common fixed point in X.

Remark 1.11 : Corollary 1.4 is a generalization of Theorem 2 of Fisher[2].

SECTION 2

Throughout this section, (X, d) is a compact metric space; $CL(X)$ is the collection of all nonempty closed subsets of X; H is the Hausdorff metric on $CL(X)$; F, G are mappings from X into $CL(X)$; P, Q are selfmaps on X;

$$S = \{(x, y)/x, y \in X,\ Fx \neq Gy,\ Px \neq Qy,\ Px \notin Fx \text{ and } Qy \notin Gy\};$$

$$T_1 = \{(x, y)/x, y \in X,\ Px \in Gy \text{ and } d(Px, Qy) = d(Qy, Gy)\};$$

and

$$T_2 = \{(x, y)/x, y \in X,\ Qy \in Fx \text{ and } d(Px, Qy) = d(Px, Fx)\}.$$

Theorem 2.1—Suppose that

(1) $Fx \subseteq Q(X)$, $Gx \subseteq P(X)$ for all x in X,

(2) either the map $x \to d(Px, Fx)$ or the map $x \to d(Qx, Gx)$ is continuous on X and

(3)
$$\begin{aligned} H(Fx, Gy) < \max \{&\beta_1(x, y)\, d(Px, Qy) + \beta_2(x, y)\, d(Px, Fx) \\ &+ \beta_3(x, y)\, d(Qy, Gy) + \beta_4(x, y)\, d(Px, Gy) \\ &+ \beta_5(x, y)\, d(Qy, Fx),\ d(Px, Qy), \\ &d(Px, Fx),\ d(Qy, Gy), \\ &\tfrac{1}{2}[d(Px, Gy) + d(Qy, Fx)]\} \end{aligned} \qquad \ldots(2.1)$$

whenever $(x,y) \in S \cap (T_1 \cup T_2)$, where β_i's are real-valued functions on $X \times X$ such that

(i) $\beta_5(x, y) \geqslant 0$, $\beta_2(x, y) + \beta_5(x, y) < 1$ and

$\beta_1(x, y) + \beta_2(x, y) + \beta_3(x, y) + 2\beta_5(x, y) \leqslant 1$ wehnever $(x, y) \in S \cap T_1$

and

(ii) $\beta_4(x, y) \geqslant 0$, $\beta_3(x, y) + \beta_4(x, y) < 1$ and

$\beta_1(x,y) + \beta_2(x, y) + \beta_3(x, y) + 2\beta_4(x, y) \leqslant 1$ whenever $(x, y) \in S \cap T_2$.

Then either F and P or G and Q have a coincidence point.

PROOF : Define $\phi : X \to R$ as $\phi(x) = d(Px, Fx)$.

Without loss of generality, we may assume that ϕ is continuous on X.

Since (X, d) is compact, there exists x_0 in X such that

$$\phi(x_0) = \inf \{\phi(x)/x \in X\}.$$

From (1) it is clear that there exist x_1, x_2 in X such that $(x_0, x_1) \in T_2$ and $(x_2, x_1) \in T_1$. Suppose that neither F and P nor G and Q have a coincidence point. Then (x_0, x_1), $(x_2, x_1) \in S$.

Taking $x = x_2$ and $y = x_1$ in inequality (2.1) and making use of the fact that $\beta_5(x_2, x_1) \geqslant 0$ and the inequalities

$$d(Qx_1, Fx_2) \leqslant d(Qx_1, Px_2) + \phi(x_2) \text{ and } \phi(x_2) \leqslant H(Fx_2, Gx_1)$$

we get

$$\phi(x_2) < \max \{[\beta_1(x_2, x_1) + \beta_3(x_2, x_1) + \beta_5(x_2, x_1)]\, d(Qx_1, Gx_1) + [\beta_2(x_2, x_1) + \beta_5(x_2, x_1)]\, \phi(x_2), d(Qx_1, Gx_1), \phi(x_2)\}.$$

Since

$$\beta_2(x_2, x_1) + \beta_5(x_2, x_1) < 1$$

and

$$(\sum_{i=1}^{3} \beta_i(x_2, x_1)) + 2\beta_5(x_2, x_1) \leqslant 1$$

it now follows that

$$\phi(x_2) < d(Qx_1, Gx_1).$$

Since inequality (2.1) and the inequalities in 3 (ii) are valid for $x = x_0$ and $y = x_1$, and $d(Qx_1, Gx_1) \leqslant H(Fx_0, Gx_1)$, it can be shown as above that

$$d(Qx_1, Gx_1) < \phi(x_0).$$

Thus we have $\phi(x_2) < \phi(x_0)$ which is a contradiction. Hence either F and P or G and Q have a coincidence point.

Remark 2.1 : Example 2.1 shows that in Theorem 2.1 one cannot ensure the existence of coincidence points for each of the pairs (F, P) and (G, Q) even if inequality (2.1) is made more stringent by dropping all the last three terms on its right-hand side and is constrained to hold whenever $Px \neq Qy$; F, G are continuous, single-valued and are commutative; P, Q are identity maps; and the B_i's are all nonnegative constants.

Example 2.1—Let $X = \{0, 1\}$ with the usual metric. Define $f, g : X \to X$ as $f0 = 1, f1 = 0, g0 = 0$ and $g1 = 1$. Then

$$|fx - gy| < \beta_1 |x - y| + \beta_2 |x - fx| + \beta_4 |x - gy|$$

whenever x, y are distinct elements of X, where $\beta_1, \beta_2, \beta_4$ are any nonnegative constants such that $\beta_1 + \beta_2 + \beta_4$ is positive.

Remark 2.2 : Example 2.2 shows that in Theorem 2.1 one cannot ensure the existence of a common fixed point for either of the pairs (F, P), (G, Q) even if inequality (2.1) is made more stringent by dropping all the last three terms on its right-hand side and its constrained to hold whenever $Px \neq Qy$; F, G are single-valued; F, G, P, Q are continuous on X; $F = G$; $P = Q$; and the β_i's are all nonnegative constants.

Example 2.2—Let $X = \{0, 1, 2\}$ with the usual metric. Define $f, P : X \to X$ as $f0 = P0 = 1, f1 = f2 = 1$ and $P1 = P2 = 2$. Clearly $f(X) \subseteq P(X)$ and

$$| fx - fy | < \beta_1 | Px - Py | + \beta_2 | Px - fx | + \beta_3 | Py - fy | + \beta_4 | Px - fy | + \beta_5 | Py - fx |$$

whenever $Px \neq Py$, where $\beta_1, \ldots, \beta_5$ are any nonnegative constants such that at least one of β_1, $\min\{\beta_2, \beta_3\}$, $\min\{\beta_4, \beta_5\}$ is positive.

Remark 2.3 : In Theorem 2.1 one cannot altogether drop the set of conditions in 3 (i) or 3 (ii) even if inequality (2.1) is made more stringent by dropping all the last three terms on its right-hand side and is constrained to hold for all x, y in X; F, G are single valued continuous maps; P, Q are continuous maps; and the β_i's are all nonnegative constants. While Example 2.3 illustrates this when $P = Q$ (each being the identity map on X). Example 2.4 does so when $F = G$.Bu t when $F = G$ and $P = Q$ it is evident that in Theorem 2.1 one can drop either 3 (i) or 3 (ii).

Example 2.3—Let $X = \{1, 2, 3, 4\}$ and d be a metric on X defined by $d(1, 2) = d(1, 4) = d(2, 3) = 1$, $d(1, 3) = d(2, 4) = d(3, 4) = \frac{3}{2}$.

Define $f, g : X \to X$ as $f1 = f3 = 2, f2 = f4 = 1, g1 = g4 = 3$ and $g2 = g3 = 4$. Then

$$d(fx, gy) < d(x, y) + d(x, gy)$$

for all x, y in X.

Example 2.4—Let $X = \{1, 2, 3, 4\}$ and d be a metric on X defined by $d(1, 2) = d(1, 4) = d(2, 3) = d(1, 3) = 1$ and $d(1, 3) = d(2.4) = d(2, 4) = \frac{3}{2}$.

Define $f, Q : X \to X$ as $f1 = 2, f2 = 3, f3 = 4, f4 = 1; Q1 = 4, Q2 = 1, Q3 = 2$ and $Q4 = 3$. Let P be the identity map on X.

Then

$$d(fx, fy) < d(Px, Qy) + d(Px, fy)$$

for all x, y in X.

Corollary 2.1—Suppose that

(1) $Fx \subseteq Q(X)$, $Gx \subseteq P(X)$ for all x in X,

(2) either the map $x \to d(Px, Fx)$ or the map $x \to d(Qx, Gx)$ is continuous on X and

(3)
$$H(Fx, Gy) < \max \Big\{ a_1(x, y)\, d(Px, Qy) + a_2(x, y)\, d(Px, Fx) + a_3(x, y)\, d(Qy, Gy) + a_4(x, y)\, d(Px, Gy) + a_5(x, y)\, d(Qy, Fx) + a_6(x, y) \frac{d(Px, Fx)\, d(Qy, Gy)}{d(Px, Qy)}$$

(equation continued on p. 267)

$$+ a_7(x, y) \frac{d(Px, Gy)\, d(Qy, Fx)}{d(Px, Qy)},$$

$$d(Px, Qy), d(Px, Fx), d(Qy, Gy),$$

$$\tfrac{1}{2}[d(Px, Gy) + d(Qy, Fx)],$$

$$\left.\frac{d(Px, Fx)\, d(Qy, Gy)}{d(Px, Qy)}, \frac{d(Px, Gy)\, d(Qy, Fx)}{d(Px, Qy)}\right\} \quad \ldots(2.2)$$

whenever $(x, y) \in S$, where a_i's are real-valued functions on $X \times X$ such that

(i) $a_5(x, y) \geq 0$, $a_2(x, y) + a_5(x, y) + a_6(x, y) < 1$ and $a_1(x, y) + a_2(x, y) + a_3(x, y) + 2a_5(x, y) + a_6(x, y) \leqslant 1$ whenever $(x, y) \in S \cap T_1$, and

(ii) $a_4(x, y) \geq 0$, $a_3(x, y) + a_4(x, y) + a_6(x, y) < 1$ and $a_1(x, y) + a_2(x, y)$ $a_3(x, y) + 2a_4(x, y) + a_6(x, y) \leqslant 1$ whenever $(x, y) \in S \cap T_2$.

Then either F and P or G and Q have coincidence point.

PROOF : Define $\beta_1, \ldots \beta_5$ from $X \times X$ to R as follows : $\beta_i(x, y) = a_i(x, y)$ for $i = 1, 4$,

$$\beta_2(x, y) = \begin{cases} a_2(x, y) \text{ if } Px = Qy \text{ or if } (x, y) \in T_2, \\ a_2(x, y) + a_6(x, y) \dfrac{d(Qy, Gy)}{d(Px, Qy)} \text{ otherwise,} \end{cases}$$

$$\beta_3(x, y) = \begin{cases} a_3(x, y) + a_6(x, y) \text{ if } (x, y) \in T_2, \\ a_3(x, y) \quad \text{otherwise,} \end{cases}$$

and

$$\beta_5(x, y) = \begin{cases} a_5(x, y) + a_7(x, y) \dfrac{d(Px, Gy)}{d(Px, Qy)} \text{ if } Px \neq Qy, \\ 0 \quad \text{otherwise.} \end{cases}$$

It can now be seen that for $Px \neq Qy$ the first term on the right-hand side of inequality (2.2) is equal to that of inequality (2.1).

Hence the right-hand side of inequality (2.2) is equal to that of inequality (2.1) when $Px \neq Qy$ and $(x, y) \in T_1 \cup T_2$. From the conditions on a_i's and inequality (2.2), it is clear that $S \cap T_1 \cap T_2 = \phi$.

It is now easy to see that the conditions on a_i's imply the corresponding conditions on β_i's in Theorem 2.1.

Hence the corollary follows.

Remark 2.4 : Theorem 2.1 remains valid if 3 (i) and 3 (ii) of the theorem are replaced by the following apparently less stringent conditions :

$$\beta_5(x_1, y_1) \geqslant 0,\ \beta_4(x_2, y_2) \geqslant 0,\ \beta_2(x_1, y_1) + \beta_5(x_1, y_1) < 1,$$

$$\beta_3(x_2, y_2) + \beta_4(x_2, y_2) < 1$$

and

$$\left[\frac{\beta_1(x_1, y_1) + \beta_3(x_1, y_1) + \beta_5(x_1, y_1)}{1-\beta_2(x_1, y_1) - \beta_5(x_1, y_1)}\right]$$

$$\left[\frac{\beta_1(x_2, y_2) + \beta_2(x_2, y_2) + \beta_4(x_2, y_2)}{1-\beta_3(x_2, y_2) - \beta_4(x_2, y_2)}\right] \leqslant 1$$

whenever $(x_1, y_1) \in S \cap T_1$ and $(x_2, y_2) \in S \cap T_2$.

In fact, in the above statement one can take either $x_1 = x_2$ or $y_1 = y_2$ according as the map $x \to d(Qx, Gx)$ or the map $x \to d(Px, Fx)$ is continuous on X. A similar remark holds in the case of Corollary 2.1 also.

Remark 2.5 : In Corollary (2.1), one can drop either 3 (i) or 3 (ii) when $F = G$ $P = Q$.

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ON RELATIVE TOPOLOGICAL DEGREE OF SET-VALUED COMPACT VECTOR FIELDS

LJ. GAJIĆ

Prirodno-mathematički fakultet, Institut za matematiku, 21000 *Novi Sad, dr Ilije Djuričića* 4, *Jugoslavija*

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The object of this paper is to define a concept of relative topological degree for a class of set-valued compact vector fields with respect to a closed convex subset in a topological vector space. Some usualy properties of this concept have been investigated too.

INTRODUCTION

The concept of relative topological degree of single-valued compact vector fields with respect to a closed convex subset in a locally convex topological vector space was introduced by Borisovich[1]. Duc *et al.*[2] generalized this concept to the set-valued case. In this paper, we propose to generalize this concept for a class of set-valued compact vector fields in not necessarily locally convex linear topological spaces.

Using the concept of relatively topological degree we define a concept of topological degree for related class of ultimately compact set-valued vector field without retraction property of Petryshyn and Fitzpatrick[7].

The paper consists of four sections. The first section sets the notations and contains basic results for finite dimensional reductions. The second section is devoted to a theory of relative topological degree in linear topological space of a class of set-valued compact vector fields. In Section 3, we shall consider the topological degree for related class of ultimately compact set-valued vector fields. The fourth and last section contains two fixed point theorems.

1. PRELIMINARY RESULTS

Let X be a Hausdorff topological vector space (HTVS) over the real numbers fields $\mathscr{R}$; $\mathscr{K}(X)$ the family of non-void closed convex subsets of X: $\mathscr{U}(0)$ the family of balanced symmetric neighbourhoods of zero; L a finite-dimensional vector subspace of X, $\mathscr{U}_L(0)$ the family of closed symmetric convex neighbourhoods of 0 in L; D a nonvoid open subset of X.

Following are some basic definitions and properties of compact vector fields in topological linear spaces. Let T be a map from topological space Y into $\mathscr{K}(X)$, let A be a subset of Y, and define $T(A) = \bigcup_{x \in A} T(x)$. T is said to be upper semi continuous (u.s c. for short) on Y iff for each $B \subset Y$ and each open $W \subset X$ with $T(B) \subseteq W$, there exists an open set V in Y such that $B \subset V$ and $T(V) \subset W$. A map T of Y into $\mathscr{K}(X)$ is said to be compact iff T is u.s.c. on Y and $T(Y)$ is relatively compact (i. e. $\overline{T(Y)}$ is compact). A compact vector field on $\bar{D}$ (to $\mathscr{K}(X)$) is a map of $\bar{D}$ into $\mathscr{K}(X)$ of the form $I - T$ ($(I - T)(x) = x - T(x)$) where T is a compact map of Y into $\mathscr{K}(X)$.

Definition 1[5]—A subset $C \subset X$ is of Z-type iff for every $U \in \mathscr{U}(0)$ there exists a $V = V(U) \in \mathscr{U}(0)$ such that:

$$\operatorname{conv}(V \cap (C - C)) \subseteq U. \qquad \ldots(1)$$

(conv = convex hull).

Remark 1: Every subset in locally convex space is of Z-type. For another, non-trivial example, see Hadžić[5].

Remark 2: It is easy to prove that for every $U \in \mathscr{U}(0)$ there exists $V = V(U) \in \mathscr{U}(0)$ such that

$$\operatorname{conv}((A + V) \cap C) \subseteq A + U \qquad \ldots(2)$$

for every convex subset A of Z-type subset C of X.

Hadžić[5] proved the following generalization of Leray-Schauder-Nagumo lemma for Z-type subsets.

Lemma 1[5]—Let C be a compact Z-type subset of X. Then for every $U \in \mathscr{U}(0)$ there exists a continuous map p of C into X and a finite set $B \subset C$ such that

$$p(C) \subset \overline{\operatorname{conv}}\, B$$

and

$$p(x) - x \in U \text{ for every } x \in C.$$

Remark: The map p in this lemma is in fact, a so called Schauder projection[3].

Proposition 1—Let Y be a Hausdorff space, X a HTVS and $T: Y \to 2^X$ compact u.s.c. mapping with $T(x)$ a closed convex non-empty subset, $T(Y) \subset C_0$, C_0 convex Z-type subset. For $U \in \mathscr{U}(0)$ let $p: \overline{T(Y)} \to L$ be a Schauder projection into a finite-dimensional linear subspace L such that:

$$p(y) - y \in V = V(U)$$

for each $y \in \overline{T(Y)}$ and for each $x \in Y$ let:

$$PT(x) = \overline{\operatorname{conv}}\, p(T(x)).$$

Then :

(a) $\overline{\text{conv}}\, PT(Y)$ and each $PT(x)$ is compact and convex subset of C_0;

(b) $PT : Y \to 2^X$ is u. s. c and finite-dimensional ;

(c) $PT(x) \subset T(x) + U$ for each $x \in Y$.

PROOF : (a) As in Dugundji[3], because $p(T(x)) \subset L$ ($\dim L < +\infty$) is compact it's convex closure $PT(x)$ is a compact and convex subset of L and C_0. For the same reason, the compactness of $\overline{T(Y)}$ implies that of $\overline{\text{conv}}\, PT(Y)$.

(b) Clearly, only that PT is u.s.c requires proof. Choose any $x \in Y$ and let $W \subset X$ be open with $PT(x) \subset W$. Since $PT(x)$ is compact there is a $U_1 \in \mathscr{U}(0)$ such that $PT(x) + U_1 \subset W$ and let $U \in \mathscr{U}(0)$ be so that $U + U \subset U_1$. Since C_0 is of Z-type there is a $V \in \mathscr{U}(0)$ so that

$$\text{conv}\,((C + V) \cap C_0) \subset C + U$$

for every convex subset C of C_0. Clearly $p\,T(x) \subseteq PT(x) \subseteq PT(x) + V$. Being the composition of two point-compact and u.s. c set functions, $x \mapsto p\,T(x)$ is also point-compact and u. s. c, so there is a neighbourhood $V(x) \in \mathscr{U}(x)$ with $pT(y) \subset PT(x) + V$ for all $y \in V(x)$. According to the choose of V we find :

$$\begin{aligned} PT(y) = \text{conv}\,(p\,T(y) \cap C_0) &\subset \overline{\text{conv}\,((PT(x) + V) \cap C_0)} \\ &\subset \overline{PT(x) + U} \subset PT(x) + U_1 \subset W \end{aligned}$$

for all $y \in V(x)$, so $PT(V) \subset W$ and because x is arbitrary PT is u.s.c.

(c) Let $z \in PT(x)$. We have $z = \sum_{i=1}^{n} \lambda_i z_i$ for suitable $z_i \in pT(x)$ and real $0 \leqslant \lambda_i \leqslant 1$ with $\sum_{i=1}^{n} \lambda_i = 1$. For each i choose $y_i \in T(x)$ so that $p(y_i) = z_i$. Then

$$p(y_i) - y_i = v_i \in V \; (i = 1, 2, \ldots, n)$$

so we have $y_i + v_i \in T(x) + V$ for each i and

$$\begin{aligned} z = \sum_{i=1}^{n} \lambda_i\, p(y_i) &= \sum_{i=1}^{n} \lambda_i (y_i + v_i) \\ &\in \text{conv}\,((T(x) + V) \cap C_0) \subset T(x) + U. \end{aligned}$$

2. A Degree Theory

Throughout this section, T is u. s. c map of $\bar{D}$ into $\mathscr{K}(X)$ and K is a closed convex subset of X such that :

(i) $H = \text{cl}\, T(K \cap \bar{D})$ is compact,

(ii) $H \subset K$

(iii) $0 \in X \setminus (I - T)(K \cap \partial D)$,

(iv) H is contained is some convex Z-type subset C_0 of X.

As in Ma's paper[6] one can prove :

Proposition 2—Let Y be a closed subset of $\bar{D}$. Then $(I - T)(K \cap Y)$ is closed. In particular, there exists a $V \in \mathscr{U}(0)$ such that :

$$V \cap (I - T)(K \cap \partial D) = \phi. \quad \text{...(3)}$$

Let $V_1 \in \mathscr{U}(0)$ be such that :

$$V_1 + V_1 \subseteq V \quad \text{...(3')}$$

where V is from Proposition 2.

Assume that $D \cap K \neq \phi$. With PT as in Proposition 1 for $U = V_1$ (V_1 as in (3')) we have that :

$$0 \notin (I - PT)(K \cap \partial D).$$

Now, let L be a finite dimensional vector subspace of X such that $p(H) \subset L$ and $L \cap K \cap D \neq \phi$. Then $L \cap K$ is closed convex subset of finite dimensional vector space L. Hence, by Tietze's theorem there exists a continuous map f of L into $K \cap L$ such that $f(x) = x$ for each $x \in L \cap K$. Let U be a relativity open subset of L such that :

$$L \cap K \cap D \subset U \subset f^{-1}(L \cap K \cap D).$$

As in Duc *et al.*[2] one can prove that $0 \notin (I - PTf)(\partial U)$ and that $PTf|_{\bar{U}}$ is a compact map on $\bar{U}$ to $\mathscr{K}(L)$ so deg $(U, 0, I - PTf)$ is defined. Whence we have the following definition:

Definition 2—Let T be u.s.c map of $\bar{D}$ into $\mathscr{K}(X)$. Suppose K is a closed convex subset of X such that (i), (ii), (iii), (iv) are satisfied.

We pose :

$$D_K(D, 0, I - T) = \begin{cases} 0 \text{ if } K \cap D = \phi, \\ \deg(U, 0, I - PTf) \text{ if } K \cap D \neq \phi, \end{cases}$$

where deg $(U, 0, I - PTf)$ is topological degree in finite dimensional vector space.

We shall say that $D_K(D, 0, I - T)$ is the degree at 0 of $I - T$ on D relative to K.

Similarly as in Duc *et al.*[2] we can prove that $D_K(D, 0, I - T)$ does not depend on the choice of PT, L, f and U so this concept is well defined.

Remark : For $K = E$ this definition is given in Gajic[4] since (iv) imply that T is uniformly finite approachable map.

Now, we shall show that $D_K(D, 0, I - T)$ enjoys usualy properties. At first

Theorem 1—Suppose $D_K(D, 0, I - T) \neq 0$. Then there exists an x in $D \cap K$ such that $x \in T(x)$.

PROOF : As in Duc *et al.*[2].

To proceed with study of the properties of D_K, we shall need the following extension of the Leray-Schauder-Nagumo lemma.

Lemma 2—Let $H_1 \subset H_2$ be two nonvoid compact subsets of convex Z-type subset $H \subset X$ and let $U \in \mathscr{U}(0)$. Then there exist two open sets $W_1 \subset W_2$ in X such that $H_1 \subset W_1$, $H_2 \subset W_2$ and a continuous map p of W_2 into $\overline{\text{conv}}\, B$, B finite subset of H_2, such that

(1) $x - p(x) \in U$ for each $x \in H \cap W_2$,

(2) $p(W_1) \subset \overline{\text{conv}}\; H_1$.

PROOF : Let be $V = V(U) \in \mathscr{U}(0)$ such that (1) (for $C = H$) is valid and $V_1 \in \mathscr{U}(0)$, $V_1 + V_1 \subset V$. Further, let $\{a_1, a_2, \ldots, a_m\} \subset H_1$ and $\{a_{m+1}, \ldots, a_n\} \subset H_2 \setminus \sum_{j=1}^{m} (a_j + \mathring{V}_1)$ be such that

$$H_1 \subset W_1 := \bigcup_{j=1}^{m} (a_j + \mathring{V}_1)$$

and

$$H_2 \setminus \bigcup_{j=1}^{m} (a_j + \mathring{V}) \subset \bigcup_{j=m+1}^{n} (a_j + \mathring{V}_1).$$

Let

$$W_2 := \Big(\bigcup_{j=1}^{m} (a_j + \mathring{V})\Big) \cup \Big(\bigcup_{j=m+1}^{n} (a_j + \mathring{V}_1)\Big). \qquad \ldots(4)$$

It seen that W_1, W_2 are open, $W_1 \subseteq W_2$, $H_1 \subset W_1$, $H_2 \subset W_2$. Let $\{q_j\}_{j=1}^{n}$ be a partition of unity for cover in (4) and

$$p(x) = \sum_{j=1}^{n} q_j(x)\, a_j,\ x \in W_2.$$

For $x \in W_2 \cap H$ we have :

$$x - p(x) = \sum_{j=1}^{n} q_j(x)(x - a_j)$$

(*equation continued on p. 274*)

$$= \sum_{\substack{j=1 \\ q_j(x)\neq 0}}^{n} q_j(x)(x - a_j)$$

$$\in \text{conv}\,(V \cap (H - H) \subseteq U.$$

As in (Duc *et al.*[2] one can prove 2).

Definition 3—Let T, K be as in Definition 2. Put $K(T, D, K, 0) = K$

$K(T, D, K, j) = \overline{\text{conv}}\, T(\bar{D} \cap K(T, D, K, j-1))$ if $(j-1)$ exists, $K(T, D, K, j) = \bigcap_{i<j} K(T, D, K, i)$ if $(j-1)$ does not exists.

If T, K are as in Definition 2 it is not difficult to prove Lemma 4, Proposition 5, Proposition 6. Theorem 3 as in Duc *et al.*[2].

Similarly as in Duc *et al.*[2] one can prove :

Theorem 2—Let F be a u s c map of $[0, 1] \times \bar{D}$ into $\mathscr{K}(X)$ with following properties :

(1) $0 \notin x - F(t, x)$ for every $(t, x) \in [0, 1] \times (K \cap \partial D)$;

(2) $H = \text{cl}\, F(J \times (K \cap \bar{D}))$ is a compact subset of K, $J = [0, 1]$;

(3) H is contained in some convex Z-type subset C_0 of X.

Put $F_t(x) = F(t, x)$ for all $(t, x) \in [0, 1] \times \bar{D}$.

Then $D_K(D, 0, I - F_t)$ is defined for each $t \in [0, 1]$ and

$$D_K(D, 0, I - F_0) = D_K(D, 0, I - F_1).$$

3. Topological Degree of Ultimately Compact Vector Fields

The concept of utimately compact vector fields was introduced by Sadovski[8]. This concept has been generalized by Petryshyn and Fitzpatrick[7] to set-valued vector fields in locally convex linear topological spaces in which closed convex sets are retracts. We shall rely on results in Section 2 on K-degree to define the concept of degree for one class of ultimately compact set-valued vector fields in general linear vector spaces without the retraction condition.

Definition 4—Let Y be a topological space and Let F be an u. s. c map of $Y \times \bar{D}$ into $\mathscr{K}(X)$ Define, for every ordinal i

$$K(F, Y \times \bar{D}, 0) = \overline{\text{conv}}\, F(Y \times \bar{D}),$$

$$K(F, Y \times \bar{D}, i) = \overline{\text{conv}}\, F(Y \times (\bar{D} \cap K(F, Y \times D, i-1)))$$
if $(i - 1)$ exists

$$K(F, Y \times \bar{D}, i) = \bigcap_{j<i} K(F, Y \times \bar{D}, j) \text{ if } (i - 1) \text{ does not exists.}$$

If no confusion can arise, we shall write K_i for $K(F, Y \times \bar{D}, i)$. If u is an ordinal strictly larger tha n the cardinal of $\overline{\text{conv}}\ F(Y \times \bar{D})$, then as is easily seen $K_i = K_u$ for every ordinal $i > u$.

If u is such an ordinal we put

$$K(F, Y \times \bar{D}, u) = K_u.$$

We shall say F is Y-ultimately compact map of $Y \times \bar{D}$ into $\mathscr{K}(X)$ if cl $F(Y \times (\bar{D} \times K(F, Y \times \bar{D}))$ is compact.

If Y is a singleton, we identify F with a map of $\bar{D}$ into $\mathscr{K}(X)$, in this case if F is Y-ultimately compact, we shall say F is ultimately compact, for short.

Definition 5—Let T be an ultimately compact map of $\bar{D}$ into $\mathscr{K}(X)$ such that $0 \notin (I - T)(\partial D)$ and $T(\bar{D})$ is contained in some convex subset of Z-type. For $K = K(T, \bar{D})$, we see from Definition 4 that K has all the properties of the K in Definition 2. Hence that $D_{K(T,\bar{D})}(D, 0, I - T)$ is defined. We say $\bar{D}_{(KT,\bar{D})}(D, 0, I - T)$ is the topological degree of the ultimately compact vector field $I - T$.

Whit T, K as in above (Definition 5), Theorems 5 and 6 of Duc *et al.*[2], Theorem 6[2] are valid and the next theorem isv alid too.

Theorem 3—Let F be a Y-ultimately compact map of $Y \times \bar{D}$ into $\mathscr{K}(X)$ such that $0 \notin x - F(t, x)$ for $(t, x) \in Y \times \partial\bar{D}$. If we suppose that $F(Y \times \bar{D})$ is contained in some convex Z-type subset then :

(1) F is ultimately compact map into $\mathscr{K}(X)$,

(2) $D_{K(F_0,\bar{D})}(N, 0, I - F_0) = D_{K(F_1,\bar{D})}(D, 0, I - F_1)$.

4. Some Fixed Point Theorems

We shall conclude this paper with fixed point theorems.

Theorem 4—Let A be a nonvoid convex not necessarily closed) Z-type subset of X and let T be a compact map of A into $\mathscr{K}(X)$ such that cl $T(A) \subset A$. Then T has a fixed point in A.

Froof : Similarly as in Duc *et al* [2] but using Proposition 1.

Theorem 5—Let B be a nonvoid convex (not necessarily closed) Z-type subset of X, let T be an ultimately compact map of B into $\mathscr{K}(X)$ such that cl $T(B) \subset B$ and $K(T, B) \neq \phi$. Then T has a fixed point.

Proof : As in Duc *et al.*[2] but using Theorem 4.

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ON FARTHEST POINT PROBLEM

B. B. PANDA

Department of Mathematics, Sambalpur University, Jyoti Vihar Burla, Sambalpur, Orissa 768017

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In this paper, it is shown that every uniquely remotal set in spaces like L^1, l^1 and L^∞ [0, 1] is a singleton. The same result is also shown to be true in subalgebras of $C(\Omega)$, where Ω is compact, and in certain infinite dimensional subspaces of a Hilbert space with a modified norm. The problem has been also explored in a subspace Y of C [0, 1] which contains congruent images of all separable Banach spaces.

1. INTRODUCTION

The present paper considers the following farthest point problem :

'Let K be a uniquely remotal set in a normed linear space X. Then is it neccessarily a singleton ?'

Recall that a nonempty bounded set K in a normed linear space X is called remotal (respectively uniquely remotal) if the map $q : X \to 2^K$ defined by $q(x) = \{y \in K : \|x - y\| = \sup_{z \in K} \|x - z\|\}$ is nonempty (respectively singleton) for every x in X. An element belonging to $q(x)$ is called a farthest point from x. The function $F_K(x) = \sup_{y \in K} \|x - y\|$ is called the farthest distance function associated with K.

Some partial and affirmative answers to the problem are known[1–9]. In the case of a Hilbert space, the problem reduces to answering another famous but unsolved problem, namely, the convexity problem of Chebyshev sets (see Klee[7]). Some general infinite dimensional spaces admitting an affirmative answer to the problem are c_0, c and $C(\Omega)$, where Ω is compact and Hausdorff. A few more cases are given by Bosznay[3] (preprint). In the present paper the problem has been solved in spaces like L^1 [0, 1], l^1 and L^∞ [0, 1]. It has been also shown that the problem admits an affirmative answer in subalgebras of $C(\Omega)$ and in certain subspaces of a Hilbert spaces with a modified norm thus improving some of the known results of Bosznay[3,4] proved under more restricted conditions. The problem has been also explored in a certain subspace Y of C [0, 1] which contains the congruent images of all separable Banach spaces.

2. MAIN RESULTS

To begin with, we consider the farthest point problem in the space L^1 [0, 1]. Accordingly, we denote

$$A_{n,i} = \left(\frac{i}{n}, \frac{i+1}{n}\right], \; i = 0, 1, \ldots, n-1$$

$$Y_n = \text{span}\,\{\chi\, A_{m,i} : 0 \leqslant i \leqslant m-1,\; 1 \leqslant m \leqslant n\}$$

and

$$Y = \text{span}\,\{\chi_{A_{m,i}} : 0 \leqslant i \leqslant m-1, \text{ and for all } m\}$$

where χ_A denotes the characteristic function of the set A.

Theorem 1—Every uniquely remotal set in $L^1\,[0, 1]$ is a singleton.

Proof : Let K be a uniquely remotal set in $L^1\,[0, 1]$. Let $x \in Y$; then $x \in Y_n$ for some n and it can be written in the form $x = \alpha_1 \chi_{B_1} + \ldots + \alpha_m \chi_{B_m}$, where B_m's $\left(1 \leqslant m \leqslant \frac{n(n+1)}{2}\right)$ are pairwise disjoint sets obtained in the usual way from $A_{m,i}$'s satisfying $\bigcup_{m=1}^{n(n+1)/2} B_m = \cup\,\{A_{m,i} : 0 \leqslant i \leqslant m-1,\; 1 \leqslant m \leqslant n\}$. Let A and B be any two disjoint sets obtained by taking union of members of the collection B_m. When n is fixed, there are only a finitely many pairs (A, B) and therefore the total number of such pairs (A, B) is countable when n ranges aver N. We shall now say that an $x \in Y$ is in $K\,(A, B)$ if

$$\int_A (x - q\,(x))\, d\mu \geqslant 0,$$

$$\int_B (x - q\,(x))\, d\mu \leqslant 0$$

and $x\,(t) = 0$ ae. on $(A \cup B)^c$. Note that one of A and B could be also empty. It will be first shown that $q\,(x_1) = q\,(x_2)$ whenever $x_1, x_2 \in K\,(A, B)$. Accordingly, define $x_3 \in Y$ as follows :

$$x_3\,(t) = \begin{cases} \max\,\{x_1\,(t), x_2\,(t)\} & \text{if } t \in A \\ \min\; x_1\,(t), x_2\,(t)\} & \text{if } t \in B \\ 0 & \text{if } t \notin A \cup B. \end{cases}$$

Then, by definition,

$$\begin{aligned} \|x_3 - q\,(x_i)\| &= \int_A (x_3 - q\,(x_i))\, d\mu + \int_B (q\,(x_i) - x_3)\, d\mu \\ &\quad + \int_{[0,1]\sim A\cup B} |\,q\,(x_i)\,|\, d\mu \\ &= \int_A (x_3 - x_i)\, d\mu + \int_A (x_i - q\,(x_i))\, d\mu + \int_B (q\,(x_i) - x_i)\, d\mu \\ &\quad + \int_B (x_i - x_3)\, d\mu + \int_{[0,1]\sim A\cup B} |\,q\,(x_i)\,|\, d\mu \\ &= \|x_3 - x_i\| + \|x_i - q\,(x_i)\| \text{ for } i = 1, 2. \end{aligned}$$

But $F_K(x_3) - F_K(x_i) \leqslant \|x_3 - x_i\|$ and this, coupled with the unique remotal property of K, implies that $q(x_i) = q(x_3)$ for $i = 1, 2$ and this is what it was claimed earlier.

Now, observing that the number of $K(A, B)$ so constructed is countable and that any $x \in Y$ belongs to at least one $K(A, B)$, we conclude that the subspace Y admits only countably many farthest points in K. But, as in Lemma 3 of Asplund[1] any one dimensional subspace L of Y is then union of a countable collection $\{L \cap q^{-1}(x)\}_{x \in K}$ (empty sets being discarded) of disjoint closed sets in L which is an impossibility since the real line (a homeomorph of L) cannot be covered by a countable collection of disjoint closed sets. Thus Y must admit a single farthest point, say, $q(x_0)$ in K.

Finally, we observe that Y is dense in $L^1[0, 1]$ and hence for any $x \in L^1[0, 1]$, there is a sequence $x_n \in Y$ such that $x_n \to x$. But $F_K(x_n) = \|x_n - q(x_0)\|$, and by the continuity of $F_K(x)$, we obtain $F_K(x) = \|x - q(x_0)\| = \|x - q(x_0)\|$. As K is uniquely remotal, this leads to $q(x) = q(x_0)$ for all $x \in L^1[0, 1]$. This implies that K must be a singleton.

In particular, when the Lebesgue measure is replaced by the counting measure, the sets A, B can be taken to be finite but disjoint subsets of N and the subspace Y is then taken to be the span of $\{e_1, e_2, \ldots e_n, \ldots\}$. We now set $x \in K(A, B)$ if

$$(x - q(x))(n) \geqslant 0 \text{ for } n \in A$$
$$< 0 \text{ for } n \in B$$

and

$$x(n) = 0 \text{ for } n \notin A \cup B.$$

The following result, a partial answer to which has been given by Theorem 2 of Bosznay[3], now follows immediately.

Theorem 2—Every uniquely remotal set in l^1 is a singleton.

Theorem 3—In $L^\infty[0, 1]$, every uniquely remotal set is a singleton.

PROOF: Let K be a uniquely remotal set in $L^\infty[0\ 1]$. For $x, y \in L^\infty[0, 1]$, define

$$A_n = \{t \in [0, 1] : |x(t) - q(x)(t)| \geqslant F_K(x) - \frac{1}{n}\},$$

and

$$B_n = \{t \in [0, 1] : |y(t) - q(y)(t)| \geqslant F_K(y) - \frac{1}{n}\}.$$

There is no loss of generality in assuming that $F_K(x) = F_K(y)$. For, if $F_K(x) > F_K(y)$, then $\lambda > 1$ can be chosen such that

$$\|\lambda y + (1 - \lambda)q(y) - q(y)\| = F_K(\lambda y + (1 - \lambda)q(y)) = \lambda F_K(y) = F_K(x).$$

We have two cases to consider.

Case I—Suppose that $\mu(A_n \cap B_n) = 0$ for some n. Then define a $z \in L^\infty[0, 1]$ by setting

$$z(t) = \begin{cases} x(t) & \text{if } t \notin B_n \\ y(t) & \text{if } t \in B_n. \end{cases}$$

Clearly, z has both $q(x)$ and $q(y)$ as farthest points and, by the unique remotal property of K, $q(x) = q(y)$.

Case II—Suppose that $\mu(A_n \cap B_n) > 0$ for all n. Denote $C_n = A_n \cap B_n$. Then the ess. sup of both $x - q(x)$ and $y - q(y)$ need only be taken on C_n to obtain $F_K(x)$ and $F_K(y)$ respectively. If $\mu(C_n) < 1$, define an L^∞-function u by putting

$$u(t) = \begin{cases} \alpha F_K(\theta) & \text{if } t \notin C_n \\ 0 & \text{if } t \in C_n \end{cases}$$

where α is so chosen that $(\alpha - 2) F_K(\theta) > F_K(x)$. Choosing a suitable $\lambda > 1$ in $x_\lambda = \lambda x + (1 - \lambda) q(x)$ satisfying $F_K(x_\lambda) = F_K(u)$, we now apply Case I to the pair (C_n, C'_n) and observing that $u - q(u)$ assumes its ess. sup norm on the set C'_n, we obtain $q(u) = q(x)$. The same way, we also obtain $q(u) = q(y)$ and, consequently, $q(x) = q(y)$.

On the other hand, if $\mu(C_n) = 1$ for all n, then with $C = \cap C_n$, we have $\mu(C) = 1$ and $|x(t) - q(x)(t)| = F_K(x)$ and $|y(t) - q(y)(t)| = F_K(y)$ for all $t \in C$. Choose an $E \subset C$ with $0 < \mu(E) < 1$ and then define

$$u(t) = \begin{cases} \alpha F_K(\theta) & \text{if } t \notin E \\ 0 & \text{if } t \in E, \end{cases}$$

where α is so chosen that $(\alpha - 2) F_K(\theta) > F_K(x)$. Again, an application of Case I to the pair (E, E') leads to $q(x) = q(y)$. Since x and y are arbitrary, K contains a single farthest point and, consequently, K must reduce to a single element.

The following is an analogue of Theorem 3 of Bosznay[3].

Theorem 4—Let H be a Hilbert space and let $\{f_n\}$ be an orthonormal sequence in H. Let $Y = [f_1, f_2, \ldots, f_n, \ldots]$. Then, for all $\epsilon > 0$, there exists a $\|.\|_\epsilon$ norm in Y such that for all $x \in Y$, $(1 - \epsilon)\|x\| \leqslant \|x\|_\epsilon \leqslant \|x\|$, and in $(Y, \|.\|_\epsilon)$, every uniquely remotal set is a singleton.

PROOF: The proof is similar to that of Theorem 3, of Bosznay[3] and we give it just for completeness.

Denote

$$Y_n = \{x \in [f_1, f_2, \ldots f_n] : \|x\| = 1\},$$

and

$$G_1 = \{x \in [f_1] : \|x\| < 1\}.$$

Suppose $\{Y_1^1, Y_2^1, \dots Y_{k(1)}^1\}$ is an $\epsilon/16$-net in G_1, and for $n > 1$, let $G_n = \{x \in Y_n : d(x, \in Y_{n-1}) > \frac{\epsilon}{4}(1 - \frac{1}{2^n})\}$, and $\{y_1^n, y_2^n, \dots y_{k(n)}^n\}$ be an $\epsilon/4^{n+1}$-net in G_n. As Y_n's are symmetric about the origin, so are the sets G_n's and therefore, the set $P = \bigcup_{n=1}^{\infty} \{y_i^n : 1 \leqslant i \leqslant k(n)\}$ can be assumed to be symmetric about the origin. Let $\{x_n\}_{n \in N}$ be an enumeration of P.

Next, let $x \in Y$ and $\|x\| = 1$. Then $x \in Y_n$ for some n and $d(y, G_{n+1}) = \frac{\epsilon}{4}(1 - \frac{1}{2^{n+1}})$ for all $y \in Y_n$. Considering the $\epsilon/4^{n+1}$-net of G_{n+1}, an element $y_i^{n+1} \in P$ can be chosen so that $\|x - y_i^{n+1}\| \leqslant \frac{\epsilon}{4} + \frac{\epsilon}{4^{n+1}} < \epsilon$, and consequently

$$\inf_n \|x - x_n\| < \epsilon. \qquad \dots(1)$$

From the fact that $Y_n \cap G_{n+1} = \phi$, and $Y_n \cup G_{n+1} \subset Y_{n+1}$, it is easy to check that

$$\min_{P_{n+1} \in P \cap Y_{n+1}} \|x - P_{n+1}\| \leqslant \inf \{\|x - y\| : y \in P \cap Y'_m\} \;\forall\; m \geqslant n + 2.$$

It now follows that $\min_n \|x - x_n\|$ exists and, therefore, (1) reduces to

$$\min_n \|x - x_n\| < \epsilon. \qquad \dots(2)$$

In view of the identity

$$< y, x_n > = \|y\| \left(1 - \frac{\left\| \frac{y}{\|y\|} - x_n \right\|^2}{2}\right) \qquad \dots(3)$$

a norm $\|.\|_\epsilon$ can be defined on Y by the formula

$$\|y\|_\epsilon = \max_n \max \{< y, x_n >, < y, -x_n >\}$$

$$= \max_n < y, x_n >, \text{ by the symmetry of } P.$$

In view of (2) and (3) and the fact that $\|x_n\| = 1$, it is easy to check that $\|.\|_\epsilon$ is a norm with the desired property. The rest follows from Theorem 3 of Asplund[1].

Now, we consider the farthest point problem in subalgebras of $C(\Omega)$, where Ω is a compact topological space. The same with Ω compact and Hausdorff and the subalgebra separating and containing the constant functions has been considered by Bosznay[4] in his Theorem 1.

Theorem 5—Every uniquely remotal set in a subalgebra $\mathscr{H}$ of $C(\Omega)$ is singleton.

PROOF : Let K be a uniquely remotal set in $\mathscr{H}$. We shall now say that an evaluation functional δ_{t_0} $(t_0 \in \Omega)$ corresponds to a farthest point $q(x)$ in K if δ_{t_0} $(x - q(x)) = \|x - q(x)\| = F_K(x)$ for some x in $\mathscr{H}$.

Assume that the evaluation functionals δ_{t_1} and δ_{t_2} correspond to farthest points $q(x)$ and $q(y)$ respectively. As usual, there is loss of generality in assuming that $F_K(x) = F_K(y)$. We then obtain

$$x(t_2) - q(y)(t_2) \leqslant y(t_2) - q(y)(t_2) = F_K(y)$$

and

$$y(t_1) - q(x)(t_1) \leqslant x(t_1) - q(x)(t_1) = F_K(x).$$

Therefore,

$$x(t_2) \leqslant y(t_2) \text{ and } y(t_1) \leqslant x(t_1).$$

Equality of any of these two will lead to $q(x) = q(y)$. So we shall assume that $x(t_2) < y(t_2)$ and $y(t_1) < x(t_1)$. Now define a function h by setting $h(t) = x(t) - y(t)$. Clearly, $h \in \mathscr{H}$, $h(t_1) > 0$ and $h(t_2) < 0$. Further define

$$\lambda(t) = \frac{h(t) - h(t_2)}{h(t_1) - h(t_2)} \cdot \frac{h(t)}{h(t_1)}, \; \chi \in \mathscr{H}$$

and

$$g(\lambda) = \frac{\lambda^2 (m - \lambda^2)^{m-1}}{(m-1)^{m-1}}, \; \|\lambda\| < \sqrt{m}$$

where m is any integer greater than 3. Clearly $g_{\min}(\lambda) = 0$, $g_{\max}(\lambda) = 1$ and $g(\lambda) \in \mathscr{H}$. The function $z(t) = g(\lambda(t))\, x(t) + (1 - g(\lambda(t)))\, y(t)$ is in $\mathscr{H}$ and has both $q(x)$ and $q(y)$ as farthest points. By the unique remotal property of K, it follows that $q(x) = q(y)$. Further, every evaluation functional corresponds to at the most one farthest point in K (for example, put $t_1 = t_2$ in the above). Thus the collection of all δ_t's $(t \in \Omega)$ correspond to a single farthest point $q(x_0)$ say, in K. Similar is the case for $-\delta_t$'s $(t \in \Omega)$. If the latter farthest point is $q(y_0)$ and if $q(x_0) \neq q(y_0)$, then $(q(x_0) + q(y_0))/2$ will admit both $q(x_0)$ and $q(y_0)$ as farthest points which is a contradiction. This complets the proof.

We note that if Ω fails to be Hausdorff, then $C(\Omega)$, with Ω compact, may fail to contain any nonconstont continuous function. Nevertheless, we have the following :

Corollary—Every uniquely remotal set in $C(\Omega)$, where Ω is compact topological space, is a singleton.

In case $\Omega = [0, 1]$ the idea that whether or not the result of Theorem 5 could be extended to every subspace of $C(\Omega)$ is quite revealing. An affirmative answer would imply that every uniquely remotal set in a separable Banach space would be a singleton. This is due to the fact that every separable Banach space is congruent with a

subspace of $C[0, 1]$ (see Holmes[6], p. 226) and, secondly, the solution to the farthest point problem remains unaffected in a congruent Banach space. This fact necessitates the study of the farthest point problem in certain special class of subspaces of $C[0, 1]$. To this end, we consider the following subspace of $C[0, 1]$. Let

$$Y = \{x \in C[0, 1] : x'(t) \text{ exists and is equal to zero for all } t \in [0, 1] \sim P, \text{ where } P \text{ is the Cantor set}\}.$$

Obviously, Lebesgue's singular function is a typical element of the space Y. It can be easily checked that Y is a complete normed linear space under the induced sup norm. The following theorem, now, generalizes the well-known result[6] that any separable Banach space is congruent with a subspace of $C[0, 1]$.

Theorem 6—Any separable Banach space is congruent with a subspace of Y.

PROOF : The proof is almost a reproduction of the same given in Holmes[6] (p. 226). The continuous function has defined from $[0, 1]$ onto $U(X^*)$ can be seen to be linear on open intervals (s_n, t_n) and, therefore, the inclusion map $i : x \to C\,U(X^*))$ followed by the congruence $T : C(U(X^*)) \to C[0, 1]$ takes X into a subspace of Y via the formula $(Tx)(t) = \langle x, h(t) \rangle \ \forall\ t \in [0, 1]$.

Theorem 7—Every uniquely remotal set K in the space Y is a singleton.

PROOF : The proof follows from the fact that Y is congruent to $C(P)$, where P is the cantor set.

Our interest lies now in congruent images in Y of separable Banach spaces. The solution to the farthest point problem in a separable Banach space would then reduce to that of an identical problem in a congruent subspace in Y. However, this remains an open problem.

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A NOTE THE ENTIRE FUNCTIONS OF L-BOUNDED INDEX AND L-TYPE

D. SOMASUNDARAM AND R. THAMIZHARASI

Department of Mathematics, Post Graduate Extension Centre, Govt. College of Engineering Campus, Salem 636011

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Growth properties and characterizations of entire functions of L-bounded index due to Lakshminarasimhan[7] are studied with the help of generalised notions of L-order and L-type for entire functions.

§1. In order to extend some of the properties of entire functions of bounded index to L-bounded index, we require the generalised notions of L-order and L-type for entire functions. Defining the L-order and L-type of entire functions in §2, we shall briefly indicate their properties analogous to those of order and type of entire functions in § 3. With the help of these notions on the growth properties of entire functions, we shall study the properties of entire functions of L-bounded index in §§ 4 and 5.

§2. *Definition* 2.1—An entire function f is said to be of finite L-order if there exists a positive constant λ such that

$$|f(z)| \leqslant \exp[rL(r)]^{\lambda} \qquad \text{.. (2 1)}$$

for $|z| = r > r_0$, where $L(r)$ is a positive continuous function increasing slowly as in the sense of 'Karamata' that is, $L(ar) \sim L(r)$ for every positive constant a.

If (2.1) is true for any λ then it is true for $\lambda' > \lambda$. The least upper bound of λ for which (2.1) is true is called the L-order of $f(z)$ and is denoted by ρ_L. The following inequalities are immediate from the definition of ρ_L.

$$M(r) = \max_{|z|=r} |f(z)| \leqslant \exp[rL(r)]^{\rho_L+\epsilon}, \ |z| = r > r_0, \epsilon > 0$$

and

$$M(r) = \max_{|z|=r} |f(z)| \geqslant \exp[rL(r)]^{\rho_L-\epsilon}$$

for an infinite number of $r \to \infty$. With the help of these two inequalities, we have the following formula for L-order

$$\rho_L = \limsup_{r\to\infty} \frac{\log\log M(r)}{\log[rL(r)]}.$$

Alternatively, $f(z)$ is of finite L-order if there exists $A > 0$ such that for $|z| = r \to \infty$

$$|f(z)| = O\,[\exp(rL(r))^A].$$

Definition 2.2—An entire function $f(z)$ of L-order ρ_L is said to be of L-type denoted by T_L, if

$$T_L = \limsup_{r\to\infty} \frac{\log M(r)}{(rL(r))^{\rho_L}}$$

exists, where $L(r)$ has the same properties as given in Definition 2.1. When $\rho_L = 1$ we say that the function is of exponetial L-type.

§3. The following properties of L-order of entire functions can be easily verified using the definition of L-order.

Proposition 1—If $f_1(z)$ and $f_2(z)$ are entire functions of L-orders p_1 and p_2 respectively and if $\rho_1 < \rho_2$ then the L-order of $F(z) = f_1(z) + f_2(z)$ is ρ_2.

Remark : This proposition fails when $\rho_1 = \rho_2$. This can be easily verified by considering $f_1(z) = \exp(z)$, $f_2(z) = -\exp(z)$ and $L(r) \equiv 1$.

The following proposition gives the relation between ρ_L and T_L.

Proposition 2—If $f(z)$ is of positive L-order ρ_L and finite L-type T_L then

$$L = \limsup_{r\to\infty} \left\{\frac{n(r)\exp(-\rho_L \log r)}{\exp(r \log L(r))}\right\} \leqslant e\,\rho_L\,T_L \qquad \text{...(3.1)}$$

$$l = \liminf_{r\to\infty} \left\{\frac{n(r)\exp(-\rho_L \log r)}{\exp(rL(r))}\right\} \leqslant \rho_L\,T_L \qquad \text{...(3.2)}$$

where $n(r)$ denotes the number of zeros in $|z| \leqslant r$.

PROOF : It is clear from the definitions of L order and L-type that

$$M(r) = O\,[\exp[rL(r)]^{\rho_L+\epsilon}]$$

and

$$M(r) = O\,[(\exp\,(T_L+\epsilon)(rL(r))^{\rho_L}]$$

so that

$$\log M(r) \leqslant (T_L+\epsilon)\,(rL(r))^{\rho_L},\ \epsilon > 0 \text{ and } r > r(\epsilon). \qquad \text{...(3.3)}$$

But from Boas[1] we have

$$N(r) = \int_0^r t^{-1}\,n(t)\,dt \leqslant \log M(r) \qquad \text{...(3.4)}$$

where $n(O) = 0$.

Using (3.4) in (3.3), we have

$$\frac{N(r)\exp(-\rho_L \log r)}{(L(r))^{\rho_L}} \leqslant T_L + \epsilon,\ r > r(\epsilon). \qquad \ldots(3.5)$$

If $n(t) > \sigma t^{\rho_L}$ for $t \geqslant t_0$, (3.5) will give

$$\frac{r^{-\rho_L}}{(L(r))^{\rho_L}} \int_0^{t_0} t^{-1} n(t)\, dt + \frac{\sigma r^{-\rho_L}}{(L(r))^{\rho_L}} \int_{t_0}^{r} t^{\rho_L - 1}\, dt \leqslant T_L + \epsilon$$

$r > r_0(\epsilon, t_0)$ and hence (3.2) follows. If $\beta > 1$, (3.5) becomes

$$n(r) \log \beta \leqslant \int_r^{\beta r} t^{-1} n(t)\, dt \leqslant \int_0^{\beta r} t^{-1} n(t)\, dt$$

that is

$$\frac{n(r)\, r^{-\rho_L}}{(L(r))^{\rho_L}} \leqslant \frac{(T_L + \epsilon)\, e^{\rho_L}}{\log \beta}$$

when $\beta = \exp(1/\rho_L)$, we get

$$\limsup_{r\to\infty} \left\{\frac{r^{-\rho_L} n(r)}{(L(r))^{\rho_L}}\right\} \leqslant e\, \rho_L\, T_L.$$

§4. *Definition* 4.1—Let $L(r)$ be as in §1. An entire function $f(z)$ is said to be of L-bounded index if there exists a positive integer k (depending on L) such that

$$\max_{0\leqslant j\leqslant k} \left\{\frac{|f^{(j)}(z)|\, L(j+2)}{j!}\right\} \geqslant \frac{|f^{(n)}(z)|\, L(n+2)}{n!} \qquad \ldots(1.2)$$

for all $z \in \mathbb{C}$ and $n = 0, 1, 2, \ldots$. The least upper bouud of such k is called the L-index of $f(z)$ and is denoted by N_L.

We shall given a characterization of a function of L-bounded index.

Theorem 1—Let $f(z)$ be an entire function of L-type T_l and L-index $N_L = p$ then $f(z)$ is of L-order $\leqslant 1$.

Proof : It has been proved[3] that

$$T_L \leqslant (p+1)\, L(p+2)/L(p+3). \qquad \ldots(4.1)$$

From the definition of L-type we know that

$$M(r) = O[\exp((T_L + \epsilon)\, rL(r))]$$

and hence we get[1] (p. 16)

$$\log M(r) \leqslant T_L + \epsilon) rL(r), \epsilon > 0, r > r(\epsilon)$$

$$\leqslant \left[\frac{(P+1) L(P+2)}{L(P+3)} + \epsilon\right] rL(r)$$

using (4.1)

$$\log \log M(r) \leqslant \log \left[\frac{(P+1) L(P+2)}{L(P+3)} + \epsilon\right] + \log [rL(r)]$$

$$\leqslant \log \left[\frac{(P+1) L(P+2)}{L(P+3)}\right] + k(\epsilon) + \log [rL(r)]$$

where $k(\epsilon) \to 0$ as $\epsilon \to 0$

$$\frac{\log \log M(r)}{\log [rL(r)]} \leqslant 1 + \frac{\log \dfrac{(P+1) L(P+2)}{L(P+3)}}{\log [rL(r)]} + \frac{k(\epsilon)}{\log [rL(r)]},$$

$$\epsilon > 0, r > r(\epsilon)\ k(\epsilon) \to 0 \text{ as } \epsilon \to 0$$

and so,

$$\limsup_{r \to \infty} \frac{\log \log M(r)}{\log rL(r)} \leqslant 1.$$

From the above inequality it is clear that L-order of $f(z) \leqslant 1$.

Theorem 2—Let $g(z)$ be an entire function and $f(z) = \exp[g(z)]$, then $f(z)$ is of L-bounded index if and only if $g(z) = \alpha z + \beta$ for some complex constants α and β.

PROOF: *Sufficient Part*: Suppose $g(z) = \alpha z + \beta$, $f'(z) = \alpha f(z)$ and so $f'(z) L(3) = KL(z) f(z)$, $K = \alpha L(3)/L(2)$ and hence $f(z)$ is of L-bounded index $\leqslant 1$.

Necessary Part: Suppose $f(z)$ is of L-bounded index, with L-order ρ_L, then using the definition of L-order in §2. We have

$$\rho_L = \limsup_{r \to \infty} \frac{\log \log M(r, f)}{\log [r\ L(r)]}.$$

By Hadamard's factorization theorem $f(z) = z^m e^{g(z)} p(z)$ where m is the order equal to the L-order of $f(z)$ and $P(z)$ is the canonical product associated with the zeros of $f(z)$. But from Theorem 1, L-order of $f(z) \leqslant 1$ and hence, the order of the polynomial $g(z) \leqslant 1$, that is $g(z) = \alpha z + \beta$ for some complex constants α and β.

Theorem 3—An entire function f is of L-bounded index if and only if for each $z \in \mathbb{C}$ there exists $r > 0$ with $a \leqslant r \leqslant d$ and a constant $M_L > 0$ such that

$$\max_{|\omega - z| = r} |f(\omega)| \leqslant M_L \min_{|\omega - z| = r} |f(\omega)|.$$

PROOF: *Necessary*: Let f be of L-index $N_L = N$ and let $d > 0$ be given. Define $d_0, d_1, \ldots, d_N$ and $r_0, r_1, \ldots, r_N$ as follows,

$$r_0 = \frac{1}{8}\,\frac{d}{1+d}$$

$$d_t = \frac{1}{4N}\; d_{t-1}\; r_{t-1}^{N}$$

$$r_t = d_t/8, \text{ for } 0 < t \leqslant N.$$

Let m be the L-index of f at $z_0 \in \mathbb{C}$. Clearly $m \leqslant N$. Now let,

$$f(z) = \sum_{0}^{\infty} a_k (z - z_0)^k$$

then

$$| a^m |\, L(m+2) \geqslant | a_j |\, L(j+2),\ j = 0, 1, 2, \ldots$$

since m is the L-index of $f(z)$ at z_0. Furthermore, let n be the smallest integer K, $0 \leqslant k \leqslant m$ such that

$$| a_k |\, K(L+2) \geqslant d_{m-k}\, | a_m |\, L(m+2)$$

such an integer k always exists. Thus, for all $z \in \mathbb{C}$ with $| z - z_0 | = r_{m-n}$

$$| f(z) | = \Big| \sum_{k=0}^{\infty} a_k (z - z_0)^k \Big|$$

$$\geqslant | a_n |\, r_{m-1}^{n} - \sum_{k \neq n} | a_k |\, r_{m-n}^{k}$$

$$\geqslant d_{m-n}\, r_{m-n}^{n}\, \frac{L(m+2)}{L(n+2)}\, | a_m | - L(m+2) \sum_{j<n} \frac{d_{n-j}\, r_{m-n}^{j}}{L(j+2)}$$

$$- L(m+2)\, | a_m | \sum_{j>n} \frac{r_{m-n}^{j}}{L(j+2)}$$

$$> \frac{| a_m |\, L(m+2)}{L(n+2)}\, r_{m-n}^{n}\, d_{m-n} \left[1 - \frac{1}{4} - \frac{L(n+2)}{L(2)} \right]$$

$$\geqslant \frac{| a_m |\, L(m+2)}{L(n+2)} \left[\frac{3}{4} - \frac{L(N+2)}{L(2)} \right] r_{m-n}^{N}\, d_{m-n} \qquad \ldots(4.2)$$

Consider

$$| f(z) | \leqslant \sum_{0}^{\infty} | a_k |\, r_{m-n}^{k} \leqslant \frac{| a_m |\, L(m+1)}{L(2)} \sum_{k=0}^{\infty} r_{m-n}^{k}$$

(equation continued on p. 289)

$$\leqslant 2 \mid a_m \mid \frac{L(m+2)}{L(2)}$$

$$\leqslant \frac{L(n+2) \min\limits_{|\omega - z| = r_{m-n}} |f(\omega)|}{\left[\frac{3}{4} - \frac{L(N+2)}{L(2)}\right] d_{m-n}\, r_{m-n}^{N}\, L(2)}$$

using (4.2)

Taking maximum over $|\omega - z| = r_{m-n}$ both sides and

$$M_L = \frac{L(N+2)}{\left[\frac{3}{4} - \frac{L(N+2)}{L(2)}\right] d_{m-n}\, r_{m-n}^{N}\, L(2)}$$

we have

$$\max_{|\omega - z| = r} |f(\omega)| \leqslant M_L \min_{|\omega - z| = r} |f(\omega)| .$$

Sufficient Part : Let $s > 0$ be given and assume the theorem is true for $d = s$, that is there exists $M_L > 0$, $a > 0$ and $a < s$ such that for each $z \in \mathbb{C}$ there exists $r > 0$ with $a \leqslant r \leqslant s$ and

$$\max_{|\omega - z| = r} |f(\omega)| \leqslant M_L \min_{|\omega - z| = r} |f(\omega)| .$$

Furthermore, choose an integer N such that $Na > s$. For $z_0 \in \mathbb{C}$ and $y_0 \in \mathbb{C}$ with $|y_0 - z_0| = 2s$ we can choose a sequence of points $y_1, y_2, \ldots, y_n$ such that $z_0, y_0, y_1, \ldots, y_N$ are collinear and $|y_{k+1} - y_k| = a$ for $k = 0, 1, 2, \ldots, N-1$. Clearly $|y_N - z_0| \leqslant s$. We have for y_j, $j = 0, 1, 2, \ldots, N$ there exists a circle $C_j = \{z : |z - z_0| = r_j\}$ with $a \leqslant r_j < s$ such that

$$\max_{z \in C_j} \{|f(z)|\} \leqslant M_L \min_{z \in C_j} \{|f(z)|\}$$

since $|y_{k+1} - y_k| = a$ and $r_j \geqslant a$, we have a sequence of circles C_{k_i}, $i = 0, 1, 2, \ldots, p$, $p < N$ such that $k_i < k_{i+1}$, $y_0 \in \{z : |z - y_{k_0}| \leqslant r_{k_0}\}$. Note that $C_{k_i} \cap C_{k_{i+1}} \neq \phi$ for $i = 0, 1, 2, \ldots, p-1$, and $C_{k_p} \cap \{z : |z - z_0| = s\} \neq \phi$. Now for $i = 0, 1, 2, \ldots, p$,

$$\max_{z \in C_{k_i}} \{|f(z)|\} \leqslant M_L^2 \min_{z \in C_{k_{i+1}}} \{|f(z)|\}.$$

Thus,

$$|f(y_0)| \leqslant \max_{z \in C_{k_0}} \{|f(z)|\}$$

$$\leqslant M_L^{2p+1} \min_{z \in C_{k_p}} \{|f(z)|\}$$

(*equation continued on p. 290*)

$$\leqslant M_L^{2N+1} \max_{|z-z_0|=s} \{ |f(z)| \}.$$

Hence for

$$\beta = M_L^{2N+1}$$

$$\max_{|z-z_0|=2s} |f(z)| \leqslant \beta \max_{|z-z_0|=s} |f(z)|$$

since s and z_0 are chosen arbitrarily from Somasundaram and Thamizharasi[5] $f(z)$ is of L-bounded index.

§5. The following theorem gives a sufficient condition for an entire function $f(z)$ to be of L-type.

Definition 5.1—Let $f(z)$ be an entire function and $z = re^{iv}$. For $p > 1$, $I(k, r)$ is defined by

$$I(k, r) = \int_0^{2\pi} |f^{(k)}(re^{iv})|^p \, dv^{1/p}.$$

Theorem 4—Let $p \geqslant 1$ and $c > 0$ be two given constants. Let $f(z)$ be entire and $z = re^{iv}$. Suppose that there exists a positive integer N such that for $k = 0, 1, 2, \ldots$

$$\sum_{j=0}^{N} \frac{I(k+j, r)}{j!} L(k+j+2) \geqslant C \sum_{j=N+1}^{\infty} \frac{I(k+j, r)}{j!} L(k+j+2)$$

holds for all z with $|z|$ sufficiently large, then $f(z)$ is of L-type and

$$T_L \leqslant 1 + 2 \log\left(1 + \frac{1}{c}\right) + \log L(N+2) + \log L(2N+2)$$

$$+ \log (2N)! - \log L(2).$$

PROOF : Consider for $r > r_0$

$$\sum_{j=0}^{N} \frac{I(k+j, r)}{j!} L(k+j+2)$$

$$\leqslant \sum_{j=0}^{2N} \frac{L(j, r)}{j!} (2N)!\, L(2N+2)\, L(j+2)/L(2)$$

$$\leqslant \left[(2N)! \, \frac{L(2N+2)}{L(2)}\right] \left[1 + \frac{1}{c}\right] \sum_{j=0}^{N} \frac{I(j, r)}{j!} L(j+2).$$

But

$$f^{(k)}(a+h) = \sum_{0}^{\infty} \frac{f^{(k+j)}(a)}{j!} h^j. \quad \text{Taking } h = e^{iv}$$

$a + h = re^{iv}$, then we have

$$f^{(k)}(a+h) = \sum_{j=0}^{\infty} f^{(k+j)}[(r-1)e^{iv}]\, e^{ijv}$$

$$\left[\int_0^{2\pi} |f^{(k)}(re^{iv})|^p\, dv\right]^{1/p} L(k+2) \leqslant \sum_{j=0}^{\infty} \left[\int_0^{2\pi} |f^{(k+j)}[(r-1)e^{iv}]|^p\, dv\right]^{1/p} \times L(k+j+2)$$

for $r > r_1 > 1 + r_0$.

That is

$$I(k,r) \leqslant \sum_{j=0}^{\infty} \frac{I(k+j, r-1)}{j!} L(k+j+2)$$

$$\leqslant \left(1 + \frac{1}{c}\right) \sum_{j=0}^{N} \frac{I(k+j, r-1)}{j!} L(k+j+2)$$

$$\leqslant (2N)!\, \frac{L(2N+2)}{L(2)} \left(1 + \frac{1}{c}\right)^2 \sum_{j=0}^{N} I(j, r-1)\, \frac{L(j+2)}{j!}$$

and so

$$\sum_{k=0}^{N} \frac{I(k,r)}{k!} L(k+2) \leqslant \frac{L(N+2)}{L(2)}\, e\, (2N)!\, L(2N+2) \left(1 + \frac{1}{c}\right)^2 \sum_{j=0}^{N} \frac{I(j, r-1)\, L(j+2)}{j!}. \qquad \ldots(5.1)$$

Put

$$\xi(r) = \sum_{k=0}^{N} \frac{I(k,r)\, L(k+2)}{k!}.$$

(5.1) gives $\xi(r) \leqslant \lambda \, \xi(r-1)$, where

$$\lambda = L(N+2)\, L(2N+2)\,(2N)!\,(1+1/c)^2 \mid L(2).$$

Extending this,

we get

$$\xi(r) < \lambda^r \, \xi(0), \; \xi(0) = \sum_{k=0}^{N} \frac{L(k+2)}{k!} = C_1.$$

Therefore,

$$\xi(r) < C_1 \, \lambda^r.$$

If

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

and

$$\mu(r) = \max_n \; | a_n | \, r^n$$

we have

$$| a_n | \, r^n \leqslant \frac{1}{2\pi} \int_0^{2\pi} | f(re^{iv}) | \, dv$$

$$\leqslant \frac{1}{2\pi} \left[\int_0^{2\pi} | f(re^{iv}) |^p \, dv \right]^{1/p} \left[\int_0^{2\pi} dv \right]^{1/q}$$

$$= (2\pi)^{-1/p} \, C_1 \, \lambda^r, \text{ where } \frac{1}{p} + \frac{1}{q} = 1.$$

That is $\mu(r) \sim \lambda^r$ but $\log \mu(r) \sim \log M(r)$. Hence .

$$\frac{\log M(r)}{rL(r)} \sim \frac{\log \mu(r)}{rL(r)} \text{ implies that}$$

$f(z)$ is of L-type $T_1 < \log \lambda$, where

$$\log \lambda = 1 + 2\left(1 + \frac{1}{c}\right) + \log L(2N+2) + \log L(N+2)$$
$$+ \log(2N)! - \log L(2).$$

Remark : The comparative growth properties of the Nevanlinna function $T(r, \rho)$ and $\log M(r, f)$ where $M(r, f) = \max_{|z|=r} | f(z) |$, are studied separately for entire functions[2] and menomorphic functions[5].

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REFLECTION OF THERMOELASTIC WAVES FROM THE STRESS-FREE INSULATED BOUNDARY OF AN ANISOTROPIC HALF-SPACE

J. N. SHARMA

Department of Mathematics, Regional Engineering College Hamirpur (HP) 177001

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The propagation of plane harmonic waves in a homogenous transversely isotropic thermally conducting elastic solid is discussed. Three types of plane waves, namely, quasi-longitudinal (QL), quasi-transverse (QT) and the mal wave (T-mode) are shown to exist. The velocities of these waves are found to depend on the angle of propagation and the frequency of the waves. The low and high frequency approximations have been obtained for the speeds of propagation and the attenuation coefficients of these waves. The reflection coefficients of these waves incident at the free surface of a transversely isotropic elastic half-space which is thermally insulated, are derived. Some special cases of reflection have also been discussed.

1. INTRODUCTION

Deresiewicz[1] studied the reflection of plane waves from a plane stress-free boundary in coupled theory of thermoelasticity and investigated the effect of boundaries on these waves. A wave reflection problem in linear coupled thermoelasticity was also considered by Beevers and Bree[2]. Sinha and Sinha[3] studied the propagation of generalized thermoelastic waves from a stress-free thermally insulated boundary of a homogeneous isotropic half-space. Sidhu and Singh[4] investigated the reflection of P and SV-waves at the free surface of a prestressed elastic half-space with orthotropic symmetry. The propagation of generalized thermoelastic waves in transversely isotropic media has been investigated by Singh and Sharma[5] and Sharma[6]. The present paper deals with the reflection of QL, QT and thermal wave (T-mode) from a stress-free thermally insulated surface of transversely isotropic elastic half-space with thermal relaxations.

2. FORMULATION OF THE PROBLEM

We consider a homogeneous transversely isotropic thermally conducting elastic medium at uniform temperature T_0. We assume that medium is transversely isotropic in such a way that the planes of isotropy are perpendicular to z-axis. We take the origin on the plane surface and z-axis normally into the medium which is thus represented by $z \geqslant 0$. We assume that the surface $z = 0$ is stress free and that there is no heat transfere across the free surface. If we restrict our analysis to plane strain parallel

to xz-plane with displacement vector $\vec{u} = (u, 0, w)$ and temperature $T(x, z, t)$, then the governing field equations of linear generalized thermoelasticity in the absence of body forces and heat sources are :

$$c_{11} u_{,xx} + c_{44} u_{,zz} + (c_{13} + c_{44}) w_{,xz} - \rho \ddot{u} = \beta_1 T_{,x} \quad \text{...(1a)}$$

$$c_{44} w_{,xx} + c_{33} w_{,zz} + (c_{13} + c_{44}) u_{,xz} - \rho \ddot{w} = \beta_3 T_{,z} \quad \text{..(1b)}$$

$$K_1 T_{,xx} + K_3 T_{,zz} - \rho C_e (\dot{T} + \tau_0 \ddot{T}) = T_0 [\beta_1 (\dot{u}_{,x} + \tau_0 \ddot{u}_{,x}) + \beta_3 (\dot{u}_{,z} + \tau_0 \ddot{u}_{,z})] \quad \text{...(1c)}$$

where

$$\beta_1 = (c_{11} + c_{12}) \alpha_1 + c_{13} \alpha_3, \; \beta_3 = 2c_{13} \alpha_1 + c_{33} \alpha_3; \quad \text{...(2)}$$

c_{ij} are the isothermal elasticities, ρ, C_e and τ_0 are respectively the density, specific heat at constant strain and thermal relaxation time; K_3, K_1 and α_3, α_1 are thermal conductivities and the coefficients of linear thermal expansions along and perpendicular to the axis of symmetry respectively. The coma notation is used for spatial derivatives and dot notation for time differentiation.

The boundary conditions are given by

$$\tau_{zz} = 0, \; \tau_{xz} = 0, \; T_{,z} = 0, \text{ at } z = 0 \quad \text{...(3)}$$

where τ_{zz} and τ_{xz} are the components of stress tensor. We define the non-dimensional quantities,

$$\left.\begin{aligned} & x' = \omega_1^* x/v_1, \; z' = \omega_1^* z/v_1, \; t' = \omega_1^* t, \; u' = \rho v_1 \omega_1^* u/\beta_1 T_0, \\ & w' = \rho v_1 \omega_1^* w/\beta_1 T_0, \; T' = T/T_0, \; \tau_0' = \omega_1^* \tau_0, \; c_1 = c_{33}/c_{11}, \; c_2 = c_{44}/c_{11}, \\ & c_3 = (c_{13}+c_{44})/c_{11}, \; \bar{K} = K_3/K_1, \; \bar{\beta} = \beta_3/\beta_1, \; \epsilon_1 = \beta^2 T_0/\rho C_e v_1^2 \end{aligned}\right\} \quad \text{..(4)}$$

where $v_1 = (c_{11}/\rho)^{1/2}$ and $\omega_1^* = C_e c_{11}/K_1$ are respectively the velocity of compressional wave in the x-direction and the characteristic frequency of the medium. Here ϵ_1 is the thermoelastic coupling constant.

Introducing the above quantities (4) in eqns. (1), we get (after supressing the dashes) :

$$u_{,xx} + c_2 u_{,zz} - \ddot{u} + c_3 w_{,xz} = T_{,x} \quad \text{..(5a)}$$

$$c_3 u_{,xz} + c_2 w_{,xx} + c_1 w_{zz} - \ddot{w} = \bar{\beta} T_{,z} \quad \text{(5b)}$$

$$T_{,xx} + \bar{K} T_{,zz} - (\dot{T} + \tau_0 \ddot{T}) = \epsilon_1 [\dot{u}_{,x} + \tau_0 \ddot{u}_{,x} + \bar{\beta} (\dot{w}_{,z} + \tau_0 \ddot{w}_{,z})] \quad \text{(5c)}$$

where τ_0 is called the thermal relaxation constant. The boundary conditions (3) become (at $z = 0$)

$$(c_3 - c_2)\, u_{,x} + c_1\, w_{,z} - \bar{\beta}\, T = 0$$

$$u_{,z} + w_{,x} = 0 \qquad \text{...(6)}$$

$$T_{,z} = 0.$$

3. Propagation of Plane Waves

For plane waves of angular frequency ω, wave number γ and phase velocity v (in general complex) incident at the free boundary $z = 0$ at an angle θ with the z-axis, we may assume

$$u = A \exp (iP_1),\ w = B \exp (iP_1),\ T = C \exp (iP_1) \qquad \text{...(7)}$$

where A, B, C are the amplitude factors and

$$P_1 = \omega\, [t - v^{-1} (x \sin\theta - z \cos\theta)] \qquad \text{.. (8)}$$

is the phase factor. Similarly, for waves reflected at $z = 0$, we assume

$$u = A \exp (iP_2),\ w = B \exp (iP_2),\ T = C \exp (iP_2) \qquad \text{...(9)}$$

where

$$P_2 = \omega\, [t - v^{-1} (x \sin\theta + z \cos\theta)] \qquad \text{...(10)}$$

is the phase factor associated with reflected waves. Substituting eqn. (7) or (9) in (5), we get

$$-(D_1 - v^2)\, A \pm c_3 \sin\theta \cos\theta\, B + i\omega^{-1} v \sin\theta\, C = 0 \qquad \text{...(11a)}$$

$$\pm\, c_3 \sin\theta \cos\theta\, A - (D_2 - v^2)\, B \mp i\, \omega^{-1} v\, \bar{\beta} \cos\theta\, C = 0 \qquad \text{...(11b)}$$

$$\epsilon_1\, \tau\, v^2 \sin\theta\, A \mp \epsilon_1\, \bar{\beta}\, \tau\, v^2 \cos\theta\, B - i\, \omega^{-1} v\, (D_3 - \tau v^2)\, C = 0 \qquad \text{...(11c)}$$

where the upper sign corresponds to the incident waves [eqn. (7)] and the lower sign corresponds to reflected waves [eqn. (8)]. D_1, D_2, D_3 and τ are given by

$$D_1 = \sin^2\theta + c_2 \cos^2\theta \qquad \text{...(12a)}$$

$$D_2 = c_2 \sin^2\theta + c_1 \cos^2\theta \qquad \text{...(12b)}$$

$$D_3 = \sin^2\theta + \bar{K} \cos^2\theta. \qquad \text{...(12c)}$$

$$\tau = \tau_0 - i\, \omega^{-1}. \qquad \text{.. (12d)}$$

The three homogeneous equations (11) in A, B, C can have a nontrivial solution only if the determinant of their coefficients vanishes, i.e.,

$$(1 + \tau_0\, \chi)\, \zeta\, (\zeta - \lambda_1^*)(\zeta - \lambda_2^*) - \chi\, D_3\, (\zeta - \lambda_1)(\zeta - \lambda_2) = 0 \qquad \text{.. (13)}$$

where

$$\zeta = v^2, \chi = i\omega, \quad \ldots(14)$$

$$\lambda_1^* = [D_1' + D_2' + \{(D_1' - D_2')^2 + 4\, c_3'^2 \sin^2\theta \cos^2\theta\}^{1/2}]/2 \quad \ldots(15a)$$

$$\lambda_2^* = [D_1' + D_2' - \{(D_1' - D_2')^2 + 4 c_3'^2 \sin^2\theta \cos^2\theta\}^{1/2}]/2 \quad \ldots(15b)$$

$$\lambda_1 = [D_1 + D_2 + \{(D_1 - D_1)^2 + 4 c_3^2 \sin^2\theta \cos^2\theta\}^{1/2}]/2 \quad \ldots(15c)$$

$$\lambda_2 = [D_1 + D_2 - \{(D_1 - D_2)^2 + 4 c_3^2 \sin^2\theta \cos^2\theta\}^{1/2}]/2 \quad \ldots(15d)$$

$$D_1' = (1 + \epsilon_1)(\sin^2\theta + C_2 \cos^2\theta) \quad \ldots(16a)$$

$$D_2' = (1 + \epsilon_1)(C_2 \sin^2\theta + C_1 \cos^2\theta) \quad \ldots(16b)$$

$$c_3' = (1 + \epsilon_1) C_3 \quad \ldots(16c)$$

and

$$C_1 = (c_1 + \epsilon_1 \bar{\beta}^2)/(1 + \epsilon_1),\ C_2 = c_2/(1 + \epsilon_1),\ C_3 = (c_3 + \epsilon_1 \bar{\beta})/(1 + \epsilon_1) \quad \ldots(17)$$

are the dimensionless isentropic elasticities.

The cubic equation (13) in ζ may be solved to obtain velocities of propagation of these waves. It may be noted that whether we take the upper or the lower sign in equations (11), i.e., whether we consider the incident or the reflected wave, we get the same three values of ζ given by equation (13). Thus, in general, in this two-dimensional model of the transversely isotropic medium, there are three types of plane waves, namely, quasi-longitudinal (QL), quasi-transverse (QT) and thermal wave (T-mode), whose phase velocities depend on the angle of incidence θ and the frequency ω and hence these waves are dispersive in character.

The equation (13) has been discussed in detail by Singh and Sharma[5] and Sharma[6].

If we write $v_j^{-1} = V_j^{-1} - iw^{-1} q_j, j = 1, 2, 3,$

then clearly V_j and q_j are speeds of propagation and the attenuation coefficients of the waves.

The low and high-frequency approximations for $\zeta_j, j = 1, 2, 3$ and hence for V_j and q_j in a more convenient form than obtained earlier[6] may be obtained as follows :

(i) *Low-frequency Approximations* ($\omega \ll 1$)

$$V_i = V_i^* \sqrt{R_i}/\cos(\phi_i/2),\ q_i = \omega \sin(\phi_i/2)/V_i^* \sqrt{R_i},\ i = 1, 2, 3$$

where

$$R_l = (A_i^2 + B_i^2)^{1/2}, \phi_l = \tan^{-1}(\pm \mid B_l/A_l \mid)$$

$$A_l = 1 - \omega^2 c_2^{(l)}, \; B_l = \omega c_1^{(l)}, V_1^* = (\lambda_i^*)^{1/2} \ldots \text{ for elastic waves}$$

$$A_3 = -\omega^2 d_2, B_3 = \omega d_1, \; V_3^* = 1, \ldots \text{ for } T\text{-mode.}$$

The coefficients $c_1^{(l)}$, $c_2^{(l)}$ d_1 and d_2 are given by

$$c_1^{(l)} = D_3 g(\lambda_i^*)/\lambda_i^* f'(\lambda_i^*), \; c_2^{(l)} = c_1^{(l)} [D_3 g'(\lambda_i^*) - \tau_0 f'(\lambda_i^*)$$

$$- c_i^{(i)} \lambda_i^* f''(\lambda_i^*)/2]/f'(\lambda_i^*)$$

$$d_1 = D_3 g(0)/f'(0), d_2 = d_1 [D_3 g'(0) - \tau_0 f'(0) - d_1 f''(0)/2]/f'(0)$$

$$f(\zeta) = \zeta(\zeta - \lambda_1^*)(\zeta - \lambda_2^*), g(\zeta) = (\zeta - \lambda_1)(\zeta - \lambda_2).$$

The signs + or − in the determination of ϕ_l are taken according as $x \sin\theta - z\cos\theta > 0$ or < 0 in (8).

(ii) *High-frequency Approximation* ($\omega \gg 1$)

$$V_l = c_l \sqrt{r_l}/\cos(\psi_l/2), q_l = \omega \sin(\psi_l/2)/c_l \sqrt{r_l}, \; i = 1, 2, 3$$

where

$$r_l = (a_i^2 + b_i^2)^{1/2}, \psi_l = \tan^{-1}(\pm \mid b_l/a_l \mid)$$

$$b_l = -r \sin\psi \, c_1^{(l)} + r^2 \sin 2\psi \, c_2^{(l)}, a_l = 1 - r \cos\psi \, c_1^{(l)} + r^2 \cos 2\psi \; c_2^{(l)}$$

$$c_l = (\lambda_l)^{1/2} \qquad \ldots \text{ for elastic waves}$$

$$a_3 = -D_3 \cos(\psi)/r + r\cos\psi \, d_1 + r^2 \cos 2\psi \, d_2 \quad \ldots \text{ for } T\text{-mode}$$

$$b_3 = D_3 \sin(\psi)/r + r \sin\psi \, d_1 + r^2 \sin 2\psi \, d_2, c_3 = 1$$

$$r = (\omega^{-2} + \tau_0^2)^{1/2}, \psi = \tan^{-1}(-1/\tau_0 \omega).$$

The coefficients $c_1^{(i)}$, $c_2^{(l)}$, d_1 and d_2 in this case are given by

$$c_1^{(i)} = f(\lambda_l)/\lambda_l D_3 g'(\lambda_l), \; c_2^{(l)} = c_1^{(l)} [f'(\lambda_l) - D_3 \lambda_l \; c_1^{(l)} g''(\lambda_l)]/D_3 g'(\lambda_l)$$

$$d_1 = \sum_{i=1}^{2} (\lambda_i^* - \lambda_l)$$

$$d_2 = 2D_3^{-1}\,[\lambda_1^*\,\lambda_2^* - \lambda_1\lambda_2 + d_1\,\{2\,(\lambda_1^* + \lambda_2^*) - \lambda_1 - \lambda_2 + 2d_1\}]$$

where primes denotes derivatives w.r.t. the arguments.

4. QL-Waves Incident at the Free Boundary

If quasi-longitudinal (QL) waves are incident at the boundary $z = 0$ of a semi-infinite transversely isotropic medium, all the three QL, QT and thermal wave (T-mode) will be generated. We may, therefore, assume that the total displacement field and temperature to be of the form

$$u = A_{i_1}\,e^{iP_1} + A_{r_1}\,e^{iQ_1} + A_{r_2}\,e^{iQ_2} + A_{r_3}\,e^{iQ_3}$$

$$w = B_{i_1}\,e^{iP_1} + B_{r_1}\,e^{iQ_1} + B_{r_2}\,e^{iQ_2} + B_{r_3}\,e^{iQ_3} \qquad \text{...(18b)}$$

$$T = C_{i_1}\,e^{iP_1} + C_{r_1}\,e^{iQ_1} + C_{r_2}\,e^{iQ_2} + C_{r_3}\,e^{iQ_3} \qquad \text{...(18c)}$$

where

$$P_1 = P_1\,(x, z) = \omega\,\{v_1\,t - (x\sin\theta_1 - z\cos\theta_1)\}/v_1 \qquad \text{...(19a)}$$

$$Q_1 = Q_1\,(x, z) = \omega\,\{v_1\,t - (x\sin\theta_1 + z\cos\theta_1)\}/v_1 \qquad \text{...(19b)}$$

are the phase factors associated with the incident and reflected QL-waves, θ_1 being the angle which these waves make with the z-axis. Similarly

$$Q_2 = Q_2\,(x, z) = \omega\,\{v_2\,t - (x\sin\theta_2 + z\cos\theta_2)\}/v_2 \qquad \text{...(20a)}$$

$$Q_3 = Q_3\,(x, z) = \omega\,\{v_3\,t - (x\sin\theta_3 + z\cos\theta_3)\}/v_3 \qquad \text{...(20b)}$$

are the phase factors of reflected QT and thermal waves respectively and θ_2, θ_3 are the angles which these waves make with the z-axis. A_i, B_i, C_i and A_r, B_r, C_r etc. are the amplitudes of incident and reflected waves respectively. Since the incident and reflected waves in (18) must satisfy the equations of motion and heat conduction equation (5), we have from the first two members in (11):

$$-\,[D_1\,(\theta_1) - v_1^2\,(\theta_1)]\,A_{i_1} + c_3\sin\theta_1\cos\theta_1\,B_{i_1} + i\,\omega^{-1}\,v_1\,(\theta_1)\sin\theta_1\,C_{i_1} = 0 \qquad \text{...(21a)}$$

$$-\,[D_1\,(\theta_j) - v_j^2\,(\theta_j)]\,A_{r_j} - c_3\sin\theta_j\cos\theta_j\,B_{r_j} + i\omega^{-1}\,v_j\,(\theta_j)\sin\theta_j\,C_{r_j} = 0$$
$$j = 1, 2, 3 \qquad \text{...(21b)}$$

and

$$c_3\sin\theta_1\cos\theta_1\,A_{i_1} - [D_2\,(\theta_1) - v_1^2\,(\theta_1)]\,B_{r_1} - i\omega^{-1}\,v_1\,(\theta_1)\,\bar{\beta}\cos\theta_1\,C_{i_1} = 0 \qquad \text{...(22a)}$$

$$-c_3 \sin\theta_j \cos\theta_j A_{r_j} - [D_2(\theta_j) - v_j^2(\theta_j)] B_{r_j} + i\omega^{-1} v_j(\theta_j) \bar{\beta} \cos\theta_j C_{r_j} = 0$$

$$j = 1, 2, 3. \qquad \text{...(22b)}$$

It may be noted that the other pairs of equations in the set (11) will give the same result as the set in eqns. (21) and (22) due to the consistency condition (13). Equations (21) and (22) may be written as

$$B_{i_1} = -F_1 A_{i_1},\ C_{i_1} = -F_1^* A_{i_1} \qquad \text{...(23a)}$$

$$B_{r_1} = F_1 A_{r_1},\ C_{r_1} = F_1^* A_{r_1} \qquad \text{...(23b)}$$

$$B_{r_2} = F_2 A_{r_2},\ C_{r_2} = F_2^* A_{r_2} \qquad \text{...(23c)}$$

$$B_{r_3} = F_3 A_{r_3},\ C_{r_3} = F_1^* A_{r_3} \qquad \text{...(23d)}$$

where

$$F_j = \bar{\beta}\{[D_1(\theta_j) - v_j^2(\theta_j)] - c_3 \sin^2(\theta_j)\}/\tan(\theta_j)\{D_2(\theta_j) - v_j^2(\theta_j) - c_3 \bar{\beta} \cos^2(\theta_j)\} \qquad \text{...(24a)}$$

$$F_j^* = \{[D_1(\theta_j) - v_j^2(\theta_j)][D_2(\theta_j) - v_j^2(\theta_j)] - c_3^2 \sin(\theta_j)\cos^2(\theta_j)\}/ i\omega^{-1} v_j(\theta_j) \sin\theta_j \{D_2(\theta_j) - v_j^2(\theta_j) - c_3 \bar{\beta}\cos^2(\theta_j)\},\ j = 1,2,3. \qquad \text{...(24b)}$$

The total displ acement field in eqns. (18)m ust also satisfy the boundary conditions (6) at $z = 0$. Making use of eqns. (18) and (23) in (6), we obtain

$$\{(c_3 - c_2)\sin(\theta_1) + c_1 F_1 \cos(\theta_1)/v_1 - \bar{\beta} F_1^*\} A_{i_1} e^{iQ_1(x,0)} + \sum_{j=1}^{3} \{(c_3 - c_2)\sin(\theta_j) + c_1 F_j \cos(\theta_j)/v_j + \bar{\beta} F_j^*\} A_{r_j} e^{iQ_j(x,0)} = 0 \qquad \text{...(25a)}$$

$$v_1^{-1}(\cos(\theta_1) + F_1 \sin(\theta_1)) A_{i_1} e^{iQ_1(x,0)} - \sum_{j=1}^{3} v_j^{-1}(\cos(\theta_j) + F_j \sin(\theta_j)) A_{r_j} e^{iQ_j(x,0)} = 0 \qquad \text{...(25b)}$$

$$F_1^* v_1^{-1} \cos(\theta_1) A_{i_1} e^{iQ_1(x,0)} + \sum_{j=1}^{3} F_j^* v_j^{-1} \cos(\theta_j) A_{r_j} e^{iQ_j(x,0)} = 0 \qquad \text{...(25c)}$$

where $v_j = v_j(\theta_j)$, $j = 1, 2, 3$ and $P_1(x, 0) = Q_1(x, 0)$.

If we write

$$v_j^{-1} = V_j^{-1} - i\omega^{-1} q_j \qquad \ldots(26)$$

then using (19.2), (20.1) and (20.2), we get

$$e^{iQ_j(x,0)} = \exp\{i\omega(V_j t - x\sin(\theta_j))/V_j\}\exp[\{-x\sin(\theta_j)q_j\}] \qquad \ldots(27)$$

where V_j is speed of propagation and q_j is attenuation coefficients of the waves.

Since the phases of the waves must be same for each value of x, we must have

$$\frac{\sin(\theta_1)}{V_1} = \frac{\sin(\theta_2)}{V_2} = \frac{\sin(\theta_3)}{V_3} \qquad \ldots(28)$$

which is the form of Snell's law for this thermally insulated stress free transversely isotropic medium.

Equations (25) on using eqns. (27) and (28) may be written as

$$(a_1 - 2\bar{\beta}\, F_1^* \exp\{-x\sin\theta_1\, q_1)\}\, A_{l_1} + a_1 A_{r_1} + a_2 A_{r_2} + a_3 A_{r_3} = 0 \qquad \ldots(29.1)$$

$$-b_1 A_{l_1} + b_1 A_{r_1} + b_2 A_{r_2} + b_3 A_{r_3} = 0 \qquad \ldots(29.2)$$

$$d_1 A_{l_1} + d_1 A_{r_1} + d_2 A_{r_2} + d_3 A_{r_3} = 0 \qquad \ldots(29.3)$$

where

$$a_j = [\{(c_3 - c_2)\sin(\theta_j) + c_1 F_j \cos(\theta_j)\}\, v_j^{-1} + \bar{\beta} F_j^*]\exp[-x\sin(\theta_j)q_j] \qquad \ldots(30.1)$$

$$b_j = -v_j^{-1}\{\cos(\theta_j) + F_j\sin(\theta_l)\}\exp\{-x\sin(\theta_j)q_j\} \qquad \ldots(30.2)$$

$$d_l = F_j^* \cos(\theta_j)\, v_j^{-1}\exp\{-x\sin(\theta_j)q_j\},\ j = 1, 2, 3. \qquad \ldots(30.3)$$

The amplitude ratios may be obtained from eqns. (23) and (29). We find

$$A_{r_1}/A_{l_1} = [\{a_1 - 2\bar{\beta} F_1^* \exp(-x\sin(\theta_1)q_1)\}(b_3 d_2 - b_2 d_3) - a_2(b_1 d_3 + b_3 d_1) + a_3(b_1 d_2 + b_2 d_1)]/\Delta \qquad \ldots(31.1)$$

$$A_{r_1}/A_{l_1} = [\{a_1 - 2\bar{\beta} F_1^* \exp(-x\sin(\theta_1)q_1)\}(b_1 d_3 - b_3 d_1) + a_1(b_1 d_3 + b_3 d_1) - 2a_3 b_1 d_1]/\Delta \qquad \ldots(31.2)$$

$$A_{r_3}/A_{i_1} = [2a_2\, b_1\, d_1 - a_2\,(b_2\, d_1 + b_1\, d_2) - \{a_1 - 2\bar{\beta}\, F_1^* \ \exp\,(-x \sin\,(\theta_1)\, q_1)\}$$
$$(b_1\, d_2 - b_2\, d_1)]/\Delta \qquad \ldots(31.3)$$

$$B_{r_1}/B_{i_1} = -\, A_{r_1}/A_{i_1} = C_{r_1}C_{i_1},\ B_{r_j}/B_{i_1} = -\, F_j\, A_{r_j}/F_1\, A_{i_1} \qquad \ldots(31.4)$$

$$C_{r_j}/C_{i_1} = -\, F_j^*\, A_{r_j}/F_1\, A_{i_1},\ j = 2,\, 3.$$

$$\Delta = a_1\,(b_2\, d_3 - b_3\, d_2) + a_2\,(b_3\, d_1 - b_1\, d_3) + a_3\,(b_1\, d_2 - b_2\, d_1). \qquad \ldots(32)$$

5. QT-Waves Incident at the Free Boundary

We assume that the incident and reflected QT-waves make angle θ_2 with the z-axis and the reflected QL-waves and thermal waves (T-mode) make angles θ_1 and θ_3 respectively with this axis. The total displacement field may be obtained by interchanging θ_1 and θ_2, v_1 and v_2 and the suffixes 1 and 2 in eqns. (18) and (19). Amplitude ratios in this case may then be obtained as in the previous section.

We obtain,

$$A_{r_1}/A_{i_2} = \{(a_2 - 2\,\bar{\beta}\, F_2^*\, \exp\,(-x \sin\,(\theta_2)\, q_2)\,(b_2\, d_3 - b_3\, d_2)$$
$$+\, a_1\,(b_1\, d_3 + b_3\, d_1) - 2a_3\, b_1 d_1\}/\Delta \qquad \ldots(33.1)$$

$$A_{r_2}/A_{i_2} = \{(a_2 - 2\,\bar{\beta}\, F_2^*\ \exp\,(-x \sin\,(\theta_2)\, q_2)\,(b_3\, d_1 - b_1\, d_3)$$
$$+ a_3\,(b_1\, d_2 + b_2\, d_1) - a_1\,(b_2\, d_3 + b_3\, d_2)\}/\Delta \qquad \ldots(33.2)$$

$$A_{r_3}/A_{i_2} = \{(a_2 - 2\,\bar{\beta}\, F_2^*\ \exp\,(-x \sin\,(\theta_2)\, q_2)\,(b_1\, d_2 - b_2\, d_1)$$
$$-\, a_2\,(b_1\, d_2 + b_2\, d_1) + 2a_1\, b_2\, d_2)/\Delta \qquad \ldots(33.3)$$

$$B_{r_j}/B_{i_2} = -\, F_j\, A_{r_j}/F_2\, A_{i_2},\ B_{r_2}/B_{i_2} = -\, A_{r_2}/A_{i_2} = C_{r_2}/C_{i_2},$$

$$C_{r_j}/C_{i_2} = -\, F_j^*\, A_{r_j}/F_2^*\, A_{i_2},\ j = 1,\, 3 \qquad \ldots(33.4)$$

where A_{i_2}, B_{i_2} and C_{i_2} are the amplitudes of incident QT-waves, and Δ is given by eqn. (32).

6. Thermal Waves (T-modes) Incident at the Free Boundary

We assume that the incident and reflected thermal waves (T-modes) make angle θ_3 with the z-axis and the reflected QL-waves and QT-waves make angles θ_1 and θ_2 respectively with this axis. The total displacement field may be obtained by interchanging θ_1 and θ_3, v_1 and v_3 and the suffices 1 and 3 in eqns. (18) and (19).

Amplitude ratios in this case may then be obtained as in the previous section. We obtain

$$A_{r_1}/A_{i_3} = \{(a_3 - 2\,\bar{\beta}\,F_3^* \exp(-x\,q_3 \sin\theta_3))\,(b_3\,d_2 - b_2 d_3) + a_3\,(b_3\,d_2 + b_2\,d_2) - 2a_2\,b_3\,d_3\}/\Delta \quad \ldots(34.1)$$

$$A_{r_2}/A_{i_3} = \{(a_3 - 2\,\bar{\beta}\,F_3^* \exp(-x\,q_3 \sin\theta_3))\,(b_1\,d_3 - b_3\,d_1) - a_3\,(b_3\,d_1 + b_1\,d_3) + 2a_1\,b_3 d_3\}/\Delta \quad \ldots(34.2)$$

$$A_{r_3}/A_{i_3} = \{(a_3 - 2\,\bar{\beta}\,F_3^* \exp(-x\,q_3 \sin\theta_3))\,(b_2\,d_1 - b_1\,d_2) + a_2\,(b_3\,d_1 + b_1 d_3) - a_1\,(b_3\,d_2 + b_2\,d_3)\}/\Delta \quad \ldots(34.3)$$

$$B_{r_j}/B_{i_3} = -\,F_j\,A_{r_j}/F_3\,A_{i_3},\; B_{r_3}/B_{i_3} = -\,A_{r_3}/A_{i_3} = C_{r_3}/C_{i_3},$$

$$C_{r_j}/C_{i_3} = -\,F_j^*\,A_{r_j}/F_3\,A_{i_3},\; j = 1, 2 \quad \ldots(34.4)$$

where A_{i_3}, B_{i_3} and C_{i_3} are the amplitudes of the incident T-modes and Δ is given by eqn. (32).

7. Special Cases

(a) *Normal Incidence*—Consider a quasi-longitudinal (QL) wave incident normally. Hence $\theta_1 = 0$ and from eqns. (28), $\theta_2 = 0 = \theta_3$. Thus F_j and F_j^* both becomes infinite for $j = 1, 2, 3$. The best way to discuss this case is to use equations (29) along with (23). We obtain the amplitudes in this case as

$$A_{r_1} = 0 = A_{r_j},\; j = 1, 2, 3\; C_{r_1} = 0 = C_{i_1}$$

$$B_{r_j} = 0 = C_{r_j},\; j = 2, 3\; B_{r_1} = B_{i_1}. \quad \ldots(35)$$

Thus a quasi-longitudinal (QL) wave is reflected as quasi-longitudinal (QL) wave without change in phase. All other waves are not reflected.

(b) *Grazing Incidence*—We consider a quasi-longitudinal (QL) wave incident at $\theta_1 = \pi/2$. In this case the ratios F_j become zero for $j = 1, 2, 3$ and $F_j^* = (D_1 - v_j^2)/i\omega^{-1}\,v_j$. The amplitudes in this case are obtained as

$$B_{i_1} = 0 = B_{r_j}, j = 1, 2, 3\; A_{r_j} = -\,A_{i_1}$$

$$A_{r_j} = C_{r_j} = 0,\; j\; 2, 3\quad C_{r_1} = 0 = C_{i_1}. \quad \ldots(36)$$

Thus there is no reflected quasi-transverse (QT) wave and thermal wave (T-mode) whereas the reflected quasi-longitudinal (QL) wave annihilates the incident quasi-longitudinal (QL) wave. Analogous results may be obtained for quasi-transverse (QT) and quasi-thermal (T-mode) waves for normal and grazing incidence.

(c) *Isotropic case*—If the transmitting material is isotropic both in elastic and thermal response then we take

$$c_{11} = c_{33} = \lambda + 2\mu,\ c_{44} = \mu,\ c_{13} = \lambda,\ \beta_1 = \beta_3 = \beta,\ K_1 = K_3 = K. \quad \ldots(37)$$

The above analysis in this case reduces to the corresponding one in the isotropic case[3].

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ON THE SPECTRA OF THERMALLY STRATIFIED TURBULENT FLOW WITH NO SHEAR

A. K. CHAKRABORTY*

Department of Mathematics, Nimta High School, Calcutta 700049

AND

H. P. MAZUMDAR

Electronic Unit, Indian Statistical Institute, Calcutta 700035

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In this paper a semi-empirical model for homogeneous turbulence in a stably stratified medium is presented. It is shown that a physical mechanism similar to 'gradient diffusion' assumption may be employed for modelling the spectral heat-flux or buyoancy term. For the terms representing inertial transfers of turbulent kinetic and temperature energy through hierarchy of eddies, Pao's [*Phys. Fluids* 8 (1965), 1063] forms are used. Analytical expressions are obtained for the turbulent kinetic and temperature energy spectra. The results as computed numerically are found helpful in resolving the differences between some existing theories.

1. INTRODUCTION

The equations for the spectral functions of turbulent velocity and temperature fluctuations in a density stratified turbulent flow with no shear, after neglecting the dissipative effects can be writen as[1]

$$\epsilon = G(k) + \beta \int_k^\infty \varphi_{wT}(k')\, dk' \qquad \ldots(1)$$

and

$$\epsilon_* = G_T(k) - 2\frac{dT}{dz} \int_k^\infty \varphi_{wT}(k')\, dk' \qquad \ldots(2)$$

where ϵ and ϵ_* are respectively, the rate of total dissipations of turbulent kinetic and temperature energy. The first terms on the right-hand side of (1) and (2) represent respectively, the inertial transfers of turbulent kinetic and temperature energy

* *Present Address :* Visiting Professor, Institute of Applied Mechanics, University of Trondheim, Norway.

through the hierarchy of eddies. The second term on the right hand side of (1) describes the contribution of the buyoancy force towards turbulent energy, while the second term on the right hand side of (2) describes the production of temperature energy due to mean temperature gradient $\frac{dT}{dz}$. $\beta (= g/T)$ is the buoyancy parameter, g is the acceleration due to gravity, T is the mean temperature, $\varphi_{wT}(k)$ is the vertical heat flux spectrum and $\int_0^\infty \varphi_{wT}(k)\, dk = \overline{w}T'$, T' being the fluctuation of temperature. It is tacitly assumed, as in Lumley[2] that the production of turbulent kinetic energy due to mean velocity shear is unimportant. Panchev and Syrakov[1] have modelled $G(k)$ and $G_T(k)$ using Heisenberg's eddy-viscosity approximation. For modelling the spectral heat-flux, they assumed that the characteristic scale of change of the total (mean plus turbulent) temperature field is a sum of the component fields[3,4]. In the present work, we use the forms for $G(k)$ and $G_T(k)$ as suggested by Pao[5] for the case when Reynolds and Peclet numbers are very high. For applicability of Pao's forms for $G(k)$ and $G_T(k)$, we assume that k must be sufficiently large so that the random velocity and temperature fields are locally homogeneous. In modelling the spectral heat-flux, we invoke a physical mechanism which is similar to the 'gradient diffusion' assumption[6].

2. Reduction of Equations (1) and (2) : Solution for the Turbulent Kinetic and Temperature Energy Spectra :

According to Pao[5], the forms for $G(k)$ and $G_T(k)$ are respectively, given by

$$G(k) = P^{-1}\, \epsilon^{1/3}\, k^{5/3}\, \varphi(k) \qquad \ldots(3)$$

and

$$G_T(k) = Q^{-1}\, \epsilon^{1/3}\, k^{5/3}\, \varphi_T(k) \qquad \ldots(4)$$

where P and Q are constants.

we may write $G(k)$ as

$$G(k) = \nu_T, \quad (\text{vorticity})^2$$

where ν_T is the turbulent transport coefficient given by

$$\nu_T = P^{-1}\, \epsilon^{-1/3}\, k^{1/3}.\, \varphi(k) \qquad \ldots(5)$$

and vorticity in this case is determined from dimensional considerations as $\epsilon^{1/3} k^{2/3}$. We model the spectral heat-flux as

$$\beta \int_k^\infty \varphi_{wT}(k')\, dk' = -\, \nu_T^*\, \beta\, \frac{d\overline{T}}{dz} \qquad \ldots(6)$$

where $\nu_T^* = \alpha \nu_T$ [cf. Panchev and Syrakov[1]], α being a dimensionless constant. It may be noticed that $\beta \dfrac{dT}{dz}$ has same dimension as that of the square of the vorticity.

In view of the relations (3), (5) and (6), eqn. (1) is reducible to

$$\epsilon = P^{-1} \epsilon^{1/3} k^{5/3} \varphi(k) - \alpha b \beta P^{-1} \epsilon^{-1/3} k^{1/3} \varphi(k) \qquad \ldots(7)$$

where $b = \dfrac{dT}{dz}$. As we confine our analysis to the case of stable stratification we assume $\dfrac{dT}{dz} > 0$. Utilizing relations (4), (5) and (6), we obtain the reduced version of eqn. (2) as

$$\epsilon_* = Q^{-1} \epsilon^{1/3} k^{5/3} \varphi_T(k) + 2 \alpha b^3 P^{-1} \epsilon^{-1/3} \varphi(k). \qquad \ldots(8)$$

We now nondimensionalize k, $\varphi(k)$ and $\varphi_T(k)$ as

$$\hat{k} = k/k_b, \; \hat{\varphi} = \varphi/\varphi_b \text{ and } \hat{\varphi}_T = \frac{\varphi T}{\varphi_T^b} \qquad \ldots(9)$$

where

$$k_b = \alpha^{3/4} \epsilon^{1/4} \epsilon_*^{-3/4} b^{3/2},$$

$$\varphi_b = \alpha^{-5/4} \epsilon^{1/4} \epsilon_*^{5/4} b^{-5/2}$$

and

$$\varphi_T^b \; \alpha^{-5/4} \epsilon^{3/4} \epsilon_*^{9/4} b^{-5/2}. \qquad \ldots(10)$$

Introducing $\hat{\varphi}$, $\hat{k}$ in equation (8), we obtain easily,

$$\hat{\varphi} = P/(\hat{k}^{5/3} - m \hat{k}^{1/3}) \qquad \ldots(11)$$

where $m = \dfrac{\beta \epsilon_*}{\epsilon b}$ is a non-dimensional parameter.

Similarly, introducing $\hat{\varphi}$, $\hat{\varphi}_T$ and $\hat{k}$ in equation (9) we obtain

$$\hat{\varphi}_T = Q \hat{k}^{-5/3} \{1 - 2 (\hat{k}^{5/3} - m \hat{k}^{1/3})\}. \qquad \ldots(12)$$

We plot log $\hat{\varphi}$ against log $\hat{k}$ for different choices of m (Fig. 1). The value of P being chosen 1.50. For the same set of values of the m, we also plot log $\hat{\varphi}_T$ against log $\hat{k}$ (Fig. 2). The value of the constant Q being taken as 0.59. As we have assumed $dT/dz > 0$, the parameter m is positive for the present case.

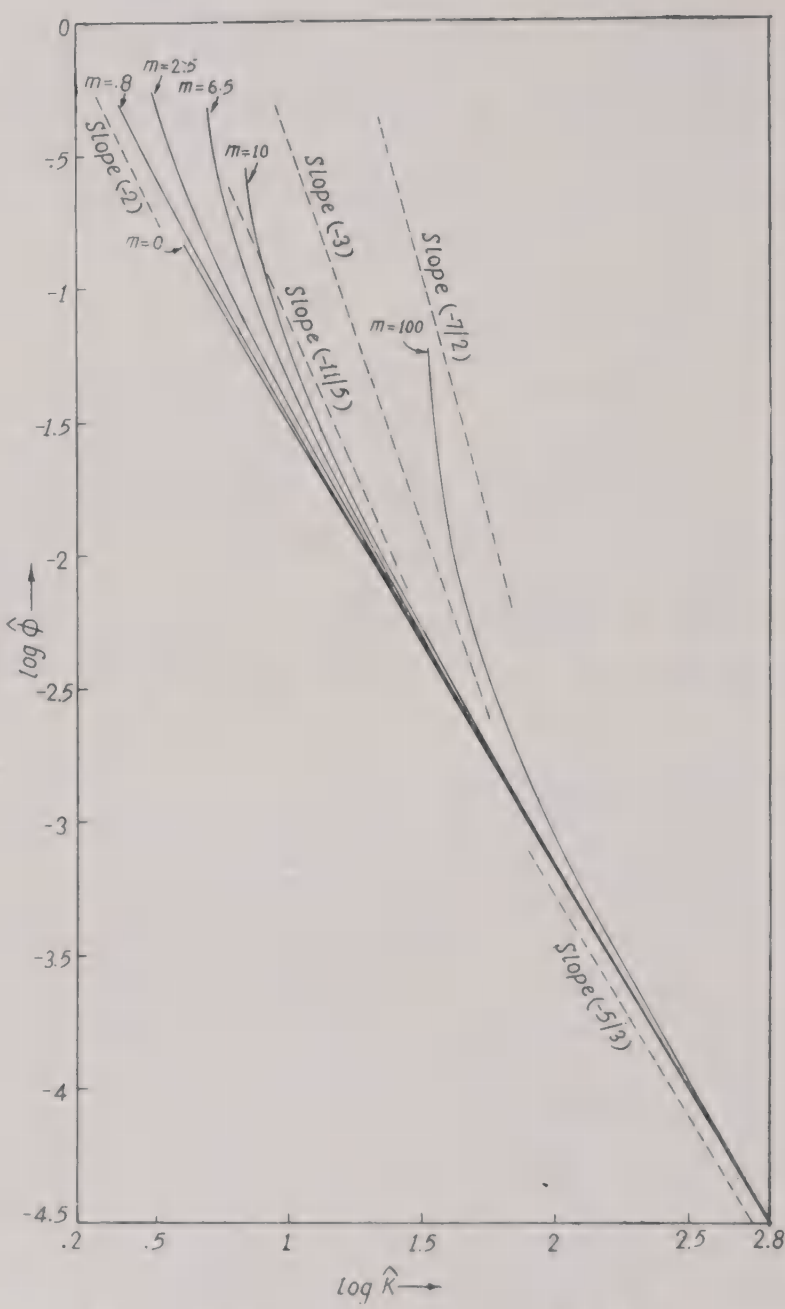

FIG. 1. Non-dimensional energy spectra of velocity.

At higher wave numbers $\hat{\varphi}$ fall off as $\hat{k}^{-5/3}$ (Fig. 1). Shur (see Reiter and Burns[7]) observed that the energy spectra of turbulence obey such $-5/3$ law (characteristic of inertial subrange) in the free atmosphere for wavelengths less than 600 m. It may be noticed that the classical intertial $-5/3$ line ($m = 0$) locates lower than the $-5/3$ asymptotes of $\hat{\varphi}$ $(\hat{k}$ (Fig. 1). For $m > 0$, there exist spectral subranges of $\hat{\varphi}$ with $\hat{k}^{-n}$ behaviour also, preceding to $-5/3$ asymptotes of $\hat{\varphi}$. At lower

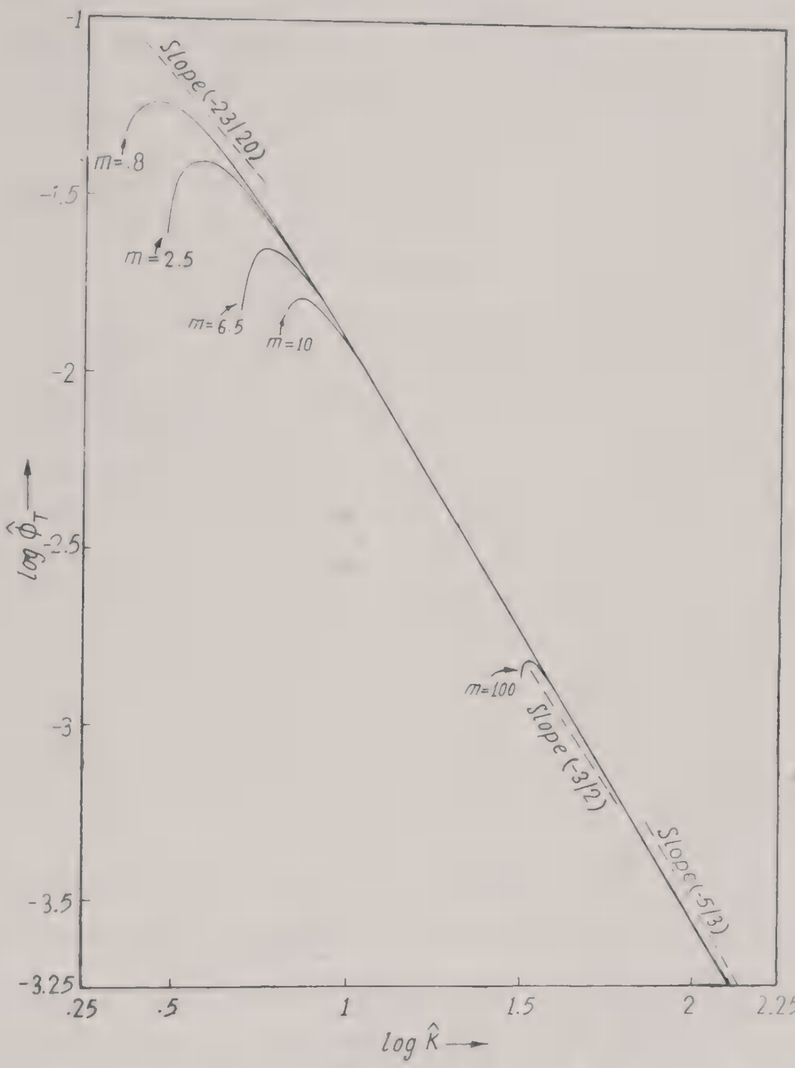

FIG. 2. Non-dimensional energy spectra of temperature.

wave numbers of the respective spectrum ranges, as treated for values 100, 10, 6.5 and 2.5 of m, n takes values 3 (Lumley[2] and Shur[8]) and 11/5 (Bolgiano[9]). In stable atmospheric environment Shur (see Reiter and Burns[7]) found n ranged between 3 and 3.2 for wavelengths larger than 700 to 800 m. According to Shur, in such low wave number regions where the modulus of the slope of the energy density curve (i.e. n) exceeds 5/3, besides dissipation of energy at the rate ϵ, there is loss of some of the energy of turbulent fluctuations due to work against the buoyancy forces of stable stratification. Lumley[2] and Bolgiano[9] forwarded theories of stably stratified turbulent flow based upon different physical premises and obtained the values of n respectively, 3 and 11/5 at low wave numbers. Lumley assumed (see text : Vinnichenko *et al.*[10]) that the rate of energy transfer in the spectrum is a function of the wave number and depends on the degee of thermal stability of the atmosphere. Bolgiano postulated (see text : Vinnichenko *et al.*[10]) that in a certain wavenumber range the rate of

dissipation of fluctuations of the specific buoyancy force has a significant effect on the shape of the spectrum. Bolgiano termed the wave-number range in which the shape of spectrum is determined by thermal stratification as "buoyancy subrange". In the present approach it is shown, depending on the appropriate choices of values of the stability parameter m (as illustrated through $m = 100, m = 10, m = 6.5$) that two different portions of one and the same spectral curve (Fig. 1) may be approximated by the power laws $\hat{\varphi} \sim \hat{k}^{-3}$ (Lumley spectrum) and $\hat{\varphi} \sim \hat{k}^{-11/5}$ (Bolgiano spectrum).

With the increase of stability, the values of n may exceed even 3. Critical examinations show that for values 100, 10, 6.5 of m, the spectra are observed with a slope of -3.5 for very short subranges (Fig. 1). Pinus and Scherbakova[11] observed in the microscale turbulence region of the upper troposphere and lower stratosphere spectra with a slope up to -3.5. It is to be noticed that as we go from higher values of m to its lower values, the -3.5, -3 and $-11/5$ subranges shrink gradually. For the case $m = .8$ all these subranges do not exist. For the temperature spectra, Monin[12] and Lumley obtained respectively the k^{-3} and k^{-1} behaviours in the buyoancy subrange. From tbe present computations, it is observable that for $m = 100$ the temperature spectrum follows a -1.5 power dependency on the wave number, in a short range of wave numbers, where buyoancy effects are predominant (Fig. 1). Bolgiano[9] obtained a $k^{-1.4}$ power law for the temperature spectrum also in the buyoancy subrange. It is noticeable that for $m = .8$, the temperature spectrum is proportional to $\hat{k}^{-1.15}$ also over a small range of wave numbers affected by the buyoancy forces. For values 100, 10, 6.5, 2.5 and .8 of m the temperature spectra exhibit $\hat{k}^{-5/3}$ fall off in the respective ranges of higher wave numbers (Fig. 1).

It may be concluded that the present simple model, based on a generalized eddy-viscosity concept and built up within the framework of similarity theory seems capable of removing some contradictions between the well known theories of Lumley and Boligiano on stably stratified turbulent flow. The model also confirms many observed spectral characteristics of stably stratified turbulent flows in the free atmosphere.

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